# Superconductors and Superfluids: The Theories That Led to the 2003 Nobel Prize in Physics

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# Superconductors and Superfluids: The Theories That Led to the 2003 Nobel Prize in Physics

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**Abstract**

The 2003 Nobel Prize in Physics was awarded to Alexei Abrikosov and Vitaly Ginzburg, for their theories related to superconductivity, and to Anthony Leggett for his theory related to superfluidity. Superconductors and superfluids have become very important over the years due to their many theoretical and experimental applications. Superconductivity was discovered by Kamerlingh Onnes in 1911, while superfluidity in $^4$He was discovered in 1938 by Pyotr Kapitsa, and independently by J.F. Allen and A.D. Misener. Superfluidity in $^3$He was not discovered until 1972 by David Lee, Douglas Osheroff and Robert Richardson. Abrikosov's theory of Type-II superconductors is the basis of the current "hot" research area of high-temperature superconductors while Leggett's theory of superfluid phase transitions has wide applications to many fields, including particle physics, cosmology, and liquid crystal physics. A brief history will be presented as well as an introduction to technical terms used in the theories. Afterwards, a descriptive analysis of the Ginzburg-Landau theory as well as the theory of type-II superconductors, and anisotropic superfluidity will be developed.

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# 1. INTRODUCTION[1]

In October 2003, the Nobel Prize was awarded to three individuals "for pioneering contributions to the theory of superconductors and superfluids".[2] These individuals are: Alexei Abrikosov, Vitaly Ginzburg, and Anthony Leggett. Abrikosov was able to describe the phenomenon of type-II superconductors; Vitaly Ginzburg and Lev Landau formulated a theory on type-I superconductors, which served as a starting point for Abrikosov's work; Anthony Leggett came up with the theory that describes the ordering and interaction of the atoms in a superfluid state. The purpose of this paper is to describe, at an introductory level, the main contributions from the 2003 Nobel laureates.

Ever since superconductors were discovered, they have become more important as they have found applications in science. Some examples of their applications include the use of superconducting magnets for: the magnetic resonance imaging (MRI) technique, which has been used in medicine for patient diagnostics, and the nuclear magnetic resonance (NMR) technique, which is often used to illuminate the structure of complex molecules. Superconducting magnets are also used in particle accelerators such as the Large Hadron Collider (LHC), which is currently under construction at CERN, in order for the large magnetic fields produced by such superconductors to bend the paths of charged particles that travel around the circumference of the accelerator. In order to fully appreciate the important components behind these techniques we must trace their beginnings, so a brief non-chronological historical excursion is now in order.

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[1] Adapted from the article by The Royal Swedish Academy of Sciences – The Information Department: *Superfluids and superconductors: quantum mechanics on a macroscopic scale,* http://www.nobel.se/physics/laureates/2003/phyadv03.pdf, posted on 7 October 2003, accessed on 14 October 2003.

[2] This was the official announcement made by the Royal Swedish Academy of Sciences.

Superconductivity was originally discovered by Kamerlingh Onnes in 1911 when he noticed that the resistivity of mercury at liquid helium temperatures disappeared completely. Superfluidity was discovered later in 1938 by Pyotr Kapitsa, who used $^4$He, and independently by A.D. Misener and J.F. Allen. Superfluidity in $^4$He is a partial manifestation of Bose-Einstein condensation, which is a phenomenon where the wave functions of individual bosonic atoms start to coalesce as the temperature of the sample is decreased and eventually become one macroscopic wave function at a certain critical temperature. $^4$He atoms can obey Bose-Einstein statistics because they have an even number of nucleons and electrons and the net spin of each atom is zero making these atoms behave like bosons. These atoms then do not have to worry about the Pauli Exclusion Principle since it only prevents fermions from occupying the same state. $^4$He atoms can then settle down into the same, lowest energy, state. On the other hand, superconductivity involves electrons, which are fermions, so it was harder to come up with an explanation of how this phenomenon is possible. In 1957, John Bardeen, Leon Cooper, and Robert Schrieffer were able to come up with an explanation for superconductivity. This explanation is now called the BCS theory and it describes superconductivity as being carried out by particles, which are now called Cooper pairs, formed by pairs of electrons. Besides explaining superconductivity, the BCS theory is also able to explain the properties of isotropic charged superfluids[3], but cannot explain the properties of anisotropic superfluids like $^3$He, which we will talk about shortly, or high temperature superconductors and heavy fermion superfluids. Electrons form pairs in order to form entities that have an integer spin and therefore behave like bosons. A

[3] Superfluids are the non-solid counterparts of superconductors. Their viscosity disappears below a certain critical temperature.

collection of Cooper pairs can then undergo Bose-Einstein condensation as the temperature is dropped. Cooper pairs can also be thought of as being structureless and isotropic in the sense that these electrons are in a spin-singlet state and they orbit each other in an s-wave state[4].

David Lee, Douglas Osheroff, and Robert Richardson discovered superfluidity in 1972 using $^3$He atoms. This is a much more complicated phenomenon than $^4$He superfluidity since $^3$He atoms have an odd number of nucleons and the net spin of these atoms is such that they behave like fermions, but as in the case of superconductivity, Helium-3 superfluidity is caused by the formation of Cooper pairs of fermions, which in this case are Helium-3 atoms. Unlike superconducting Cooper pairs, $^3$He Cooper pairs have a net non-zero orbital angular momentum and this is the reason why they are characterized as having internal degrees of freedom. They are also characterized as anisotropic particles in a spin triplet state, with the Cooper pair constituents orbiting each other in a p-state.

There are a couple of important concepts that will be relevant throughout the rest of our discussion of superconductors and superfluids: the *order parameter* and the *spontaneously broken symmetry*. The former was originally introduced by the Russian physicist Lev Landau in his theory of second order phase transitions. The order parameter is a quantity that is zero above a critical temperature $T_c$ and nonzero below this temperature. More specifically, it is a complex number $\Psi$ that has two components: an amplitude $|\Psi|$ and a phase or gauge $\phi$. The amplitude is a measurement of the density of Cooper pair particles and its value is given by their probability amplitude. The phase

[4] Cooper pairs in "s- and p-wave states" just means that their orbits have the same form as s- and p-orbitals.

helps us determine whether a system of particles can be characterized as having broken gauge symmetry[5]. When a system is above a certain critical temperature, it is said to be rotationally invariant in both spin and orbital space under a certain gauge transformation $\phi \rightarrow \phi'$. When the temperature of the system drops below the critical temperature, the system opts for a particular value of the gauge. This is known as broken gauge symmetry and it is what happens with certain materials when they become superfluids or superconductors below a certain critical temperature. The concept of the order parameter could be better understood through an example. In the theory of ferromagnetism, we can choose the order parameter to be the spontaneous magnetization of a sample. It would have a value of zero in its paramagnetic state, which is characterized by the random direction of the spins of the individual atoms of the sample, and it would have a nonzero value in the magnetically ordered ferromagnetic state, which is characterized by a preference in the spin direction of the individual atoms. The symmetry of the ferromagnet is then said to be broken under spin rotation.

The British physicist Anthony Leggett was awarded the Nobel prize for making the theoretical discovery regarding the possibility of spontaneously broken symmetries in anisotropic superfluids such as $^3$He. The reason that this is possible is that anisotropic superfluids have a much more complicated form of their order parameter than that of isotropic superfluids. The order parameter of anisotropic superfluids has at least 18 components.[6] The reason for this complication is the fact that the Cooper pairs have a

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[5] It is important to note the difference between spontaneously broken symmetry and broken gauge symmetry. The former refers to a symmetry in both spin and orbital angular momenta in a system of particles while the latter refers to a symmetry in the phase of particles.

[6] The $^3$He atoms have three substates for the spin ($S_z$=0, ±1) as well as for the relative orbital angular momentum ($l_z$=0, ±1). The wave function Ψ that describes a $^3$He Cooper pair therefore has a total of 3×3=9 components, where each component represents a particular spin and orbital angular momentum

nonzero orbital angular momentum and that they are in a spin triplet state. These factors lead to the possibility of broken rotational symmetry in spin space and broken orbital rotation symmetry in orbital space for anisotropic superfluids.

There can be a combination of broken symmetries in both orbital and spin space (Fig. 1). Leggett called this phenomenon spontaneously broken spin-orbit symmetry or simultaneously broken continuous symmetries. This phenomenon leads to superfluid properties or phases that have long range order that would not be understood if we were to combine the properties of materials that have individual broken symmetries. In other words, if we were to mix the sets of arrows from Fig. 1b and Fig. 1c, we would not get the state in Fig. 1d. Leggett was also able to show that the phenomenon of spontaneously broken spin-orbit symmetry leads to properties in the A-phase of superfluid $^3He$ that are exactly the same as a state that was already known, called the ABM state. This is also the case with the B-phase of superfluid $^3He$ and the BW state. The ABM state and the A-phase and the BW state and the B-phase are thus synonymous to each other.[7]

configuration. Each of the substates would have a related complex-valued amplitude $\psi$. D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3*, (Taylor and Francis, London, 1990), p. 9.

[7] All of the superfluid $^3He$ phases will be discussed in more detail in Sec. 4.

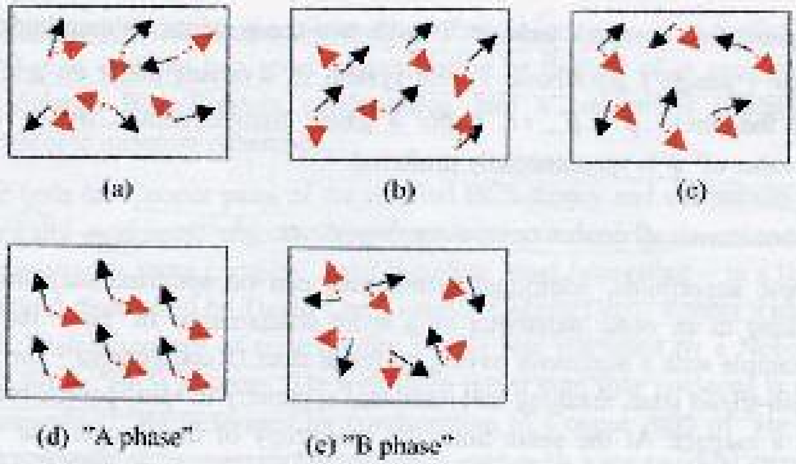


**Figure 1.**

Representation of the possible states of a two-dimensional model of a liquid made of particles that have two internal degrees of freedom. These are: the spin (black arrow) and orbital angular momentum (red arrow). (a) The disordered state. In this case, the system of particles is said to be invariant under rotations in spin and orbital space with no long-range order. The states in (b)-(e) have long-range order. (b) This state has broken rotational symmetry in spin space (corresponds to a ferromagnetic liquid). (c) This state has broken rotational symmetry in orbital space (corresponds to the configuration of a liquid crystal). (d) The spin and orbital space rotational symmetries are separately broken (corresponds to the A phase of superfluid $^3He$). (e) The broken symmetry related to the relative orientation between the orbital and spin degrees of freedom (corresponds to the B phase of superfluid $^3He$). It was Anthony Leggett who first introduced the term "spontaneously broken spin orbit symmetry", which can be seen in the (d) and (e) states. (Adapted from the Royal Swedish Academy of Sciences (op cit, p. 4)).

In order to understand superconductivity, we first need to review some general characteristics of pure and impure metals. We can think of the atoms in a metal as being distributed in a perfect and periodic lattice and the conduction electrons as traveling plane waves within the atomic lattice. The conduction electrons, being plane waves, have the characteristic that they can travel through the metal unimpeded by the atomic lattice if the lattice is perfectly periodic. Any fault in the periodicity of the lattice will cause the conduction electrons to be scattered. This scattering is what leads to electrical resistance. There are two factors that will spoil the perfect periodicity of any metal: the thermal vibrations of the constituent atoms, which will be present above absolute zero and are more pronounced as the temperature of the metal is increased, and any impurities or foreign atoms and defects present in the sample.[8] If we imagine passing some current through a metal and we measure its resistance as we change its temperature, we would notice that, for a pure sample, its resistance would decrease as we decrease its temperature and it would eventually go down to zero at absolute zero. For an impure metal, like most metals in the real world, the conduction electrons would undergo impurity scattering besides the scattering caused by the thermal vibrations of the atoms. This impurity scattering is more or less independent of temperature and will be present even at the lowest temperatures. This is why, as the temperature of an impure metal is decreased, the resistance will decrease like it did with a pure metal, but there will be some residual resistance even at absolute zero (Fig. 2).

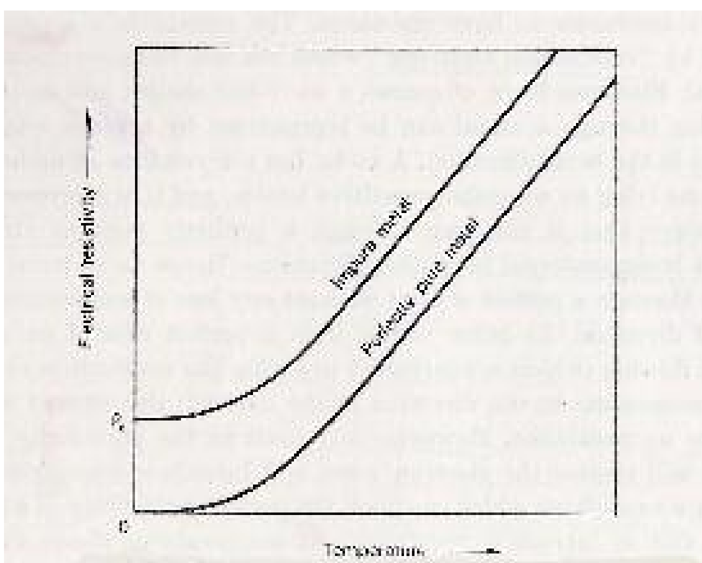


FIG. 2. Relationship between the metallic resistance and temperature. (Adapted from A.C. Rose-Innes and E. H. Rhoderick (op cit, p. 4)).

[8] A.C. Rose-Innes and E.H. Rhoderick, *Introduction to Superconductivity*, (Pergamon Press, London, 1969), p. 4.

If the metallic sample we were considering earlier were a superconductor, we would notice a very different behavior in its resistance as its temperature is lowered. The metal's resistance will decrease in the same way as for the pure and impure metals that were mentioned earlier, but when the temperature reaches a certain critical value $T_c$, the metal's resistance would suddenly vanish and it would remain that way down to absolute zero (Fig. 3). The phenomenon of superconductivity arises from the formation of Cooper pairs, which have condensed to their ground state and move coherently across the metal. [9]

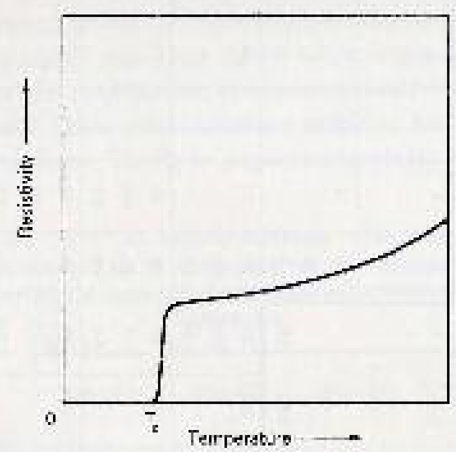


FIG. 3. The sudden loss of resistance of a superconductor at the critical temperature $T_c$. (Adapted from Rose-Innes (op cit. p. 5)).

Diamagnetism is a property of superconductors, so they will expel any external magnetic field that is applied to them besides exhibiting zero resistance. The diamagnetic phenomenon is also known as the Meissner effect.

There are two types of the Meissner effect: the complete and partial Meissner effect. The former occurs when the transition from the superconducting state to the

[9] Cooper pair formation is due to electron-phonon interactions. A conduction electron interacts with the atomic lattice of the metal and produces a phonon, which is a quantized vibrational motion of the atomic lattice. This phonon then interacts with another electron and this interaction turns out to be an attractive one if the difference in the energy of an electron before and after the interaction is less than hν, where h is Planck's constant and ν is the frequency of the phonon. If the attractive force due to this interaction is greater than the Coulomb repulsion between the electrons then these will form a Cooper pair. (A.C. Rose-Innes and E.H. Rhoderick (op cit, pp. 118-120))

normal state is discontinuous at a certain critical value of the external magnetic field ($H_c$). In this case, an applied magnetic field will be expelled from the superconductor at all values of the external magnetic field below the critical value. This is a property of type-I superconductors. [10] On the other hand, the partial Meissner effect occurs when there is a gradual transition from the superconducting state to the normal state. In this case, an external magnetic field will only be expelled from the superconductor below the lower critical magnetic field value $H_{c1}$, which is designated as the "Meissner phase" in Fig. 4. Above the lower critical magnetic field value and below the upper critical magnetic field value $H_{c2}$, the superconductor is said to be in the mixed state, or "Shubnikov phase" in Fig. 4, where the external magnetic field penetrates into the superconductor. This is a property of type-II superconductors.

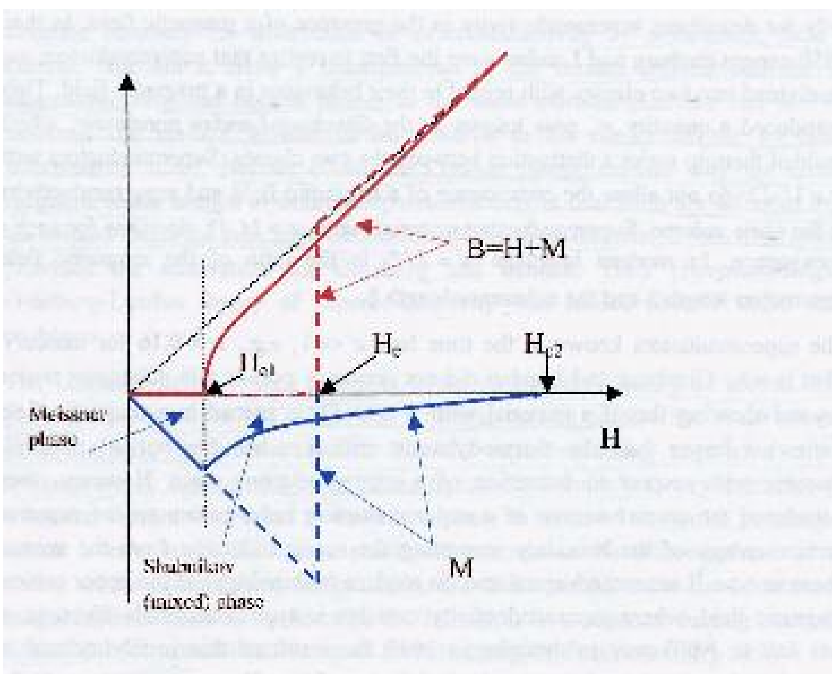


**Figure 4.** The complete (dashed lines) and partial (full lines) Meissner Effect. The red plot represents the induced field B and the blue graph represents the magnetization M in a superconducting sample, both as a function of an applied magnetic field H. (Adapted from the Royal Swedish Academy of Sciences (op cit. p. 5)).

[10] The property that gives rise to type-I and type-II superconductivity will be explained in section 3.1 "The concept of the surface energy".

The next Nobel laureate, the Russian physicist Alexei Abrikosov, published a paper on type-II superconductors in 1957. In this paper, he described the mixed state as a phenomenon characterized by magnetic field vortices that can be mathematically described in terms of the order parameter $\Psi$, the square of which describes the density of superconducting electrons in the material. These vortices only exist between the lower and upper critical magnetic fields $H_{c1}$ and $H_{c2}$. They tend to repel each other and settle into certain specific patterns, which include square, triangular, and hexagonal shapes. As the magnitude of the external magnetic field is increased, the vortices move closer to each other until, at the upper critical magnetic field value, they overlap and the order parameter is suppressed.

Two concepts that are very important in helping us understand the nature of the differences between type-I and type-II superconductors are the penetration depth[11] $\lambda$ and the coherence length $\xi$. As was stated earlier, superconductors tend to expel external magnetic fields, but these fields are not blocked out completely at the surface of the superconductor. The external magnetic field is able to penetrate a distance about equal to the penetration depth into the superconductor. At the surface of the superconductor, the number of superconducting electrons increases and reaches a constant value over a distance about equal to the coherence length into the superconducting material (Fig. 5).

[11] As shown by the London theory, which will soon be developed in Section 2.1, the value of the magnetic flux density will decrease exponentially inside a superconductor when an external magnetic field is applied parallel to its surface. It will fall to 1/e of its magnitude at a distance inside the superconductor equal to the London penetration depth $\lambda_L$, sometimes referred to as just the penetration depth $\lambda$. (A.C. Rose-Innes and E.H. Rhoderick (op cit, pp. 35-36))

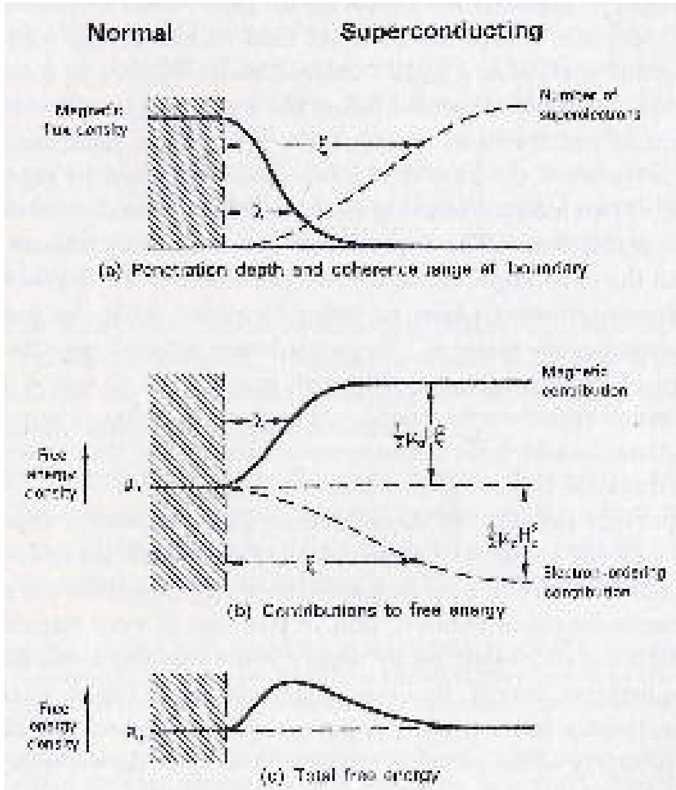


FIG. 5 . Plots depicting the emergence of positive surface energy. (Adapted from Rose-Innes (op cit, p. 80)).

Alexei Abrikosov was able to come up with his theory of type-II superconductors by an analysis of the Ginzburg-Landau equations for superconductors, which were discovered by the Russian physicists Vitaly Ginzburg, the third 2003 Nobel laureate, and Lev Landau in 1950. The Ginzburg-Landau (GL) theory was originally motivated by the need to explain the sudden destruction of superconductivity by a certain value of an applied magnetic field and an electric current. The GL theory is phenomenological, which means that it makes some ad-hoc assumptions in order to explain certain observed phenomena. The theory is able to predict whether a sample will be a type-I or type-II superconductor based on a quantity, which is now called the Ginzburg-Landau parameter $\kappa$, given by the ratio of the penetration depth to the coherence length of the sample: $\kappa=\lambda/\xi$.

If the Ginzburg-Landau parameter is less than $2^{-1/2}$, magnetic fields and superconductivity cannot coexist, so the superconductor is of type-I. On the other hand, if $\kappa > 2^{-1/2}$, magnetic fields and superconductivity can coexist and the superconductor is of type-II (Fig. 6)[12].

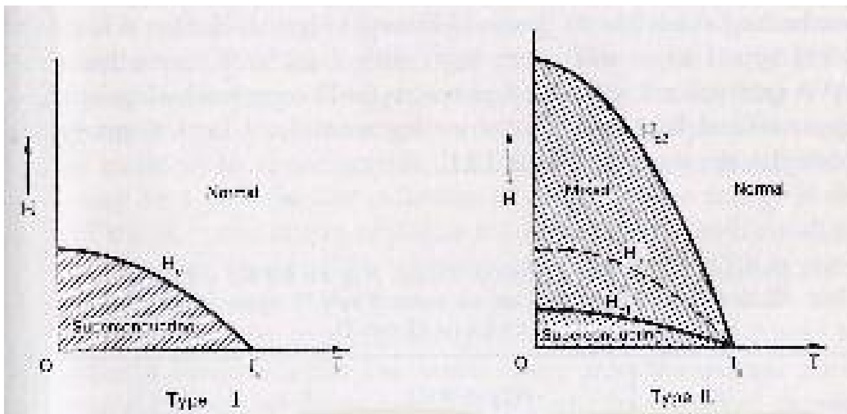


FIG. 6. The phase diagrams (with applied magnetic field H versus temperature T) of type I and type II superconductors. (Adapted from Rose Innes (op cit, p. 183)).

## 2. THE GINZBURG-LANDAU THEORY

### 2.1 Motivation for the theory: the London theory

One of the main motivations behind the development of Vitaly Ginzburg's and Lev Landau's theory of superconductivity was to introduce corrections to the phenomenological theory developed by Fritz and Heinz London. In other words, the London brothers had introduced some ad-hoc assumptions into their theory in order to describe certain observed phenomena. The London brothers already knew about the Meissner effect and the fact that, as a material becomes superconducting at the critical temperature $T_c$, surface currents will suddenly develop and circulate in such a way as to

[12] Most of the concepts mentioned in the figure will (hopefully) make more sense later on.

produce a magnetic field that will cancel all magnetic fields that would otherwise be present inside the material (Fig. 7).[13] Since the London brothers knew that the magnetic field inside the superconductor should be equal to zero, they changed certain equations that predicted a constant for the magnitude of the magnetic field inside a superconductor to make sure that the magnetic field is zero. The resulting London equations should then be treated as approximations due to their phenomenological nature as opposed to, say, Maxwell's equations, which are considered to be exact equations that reflect unbreakable physical laws.

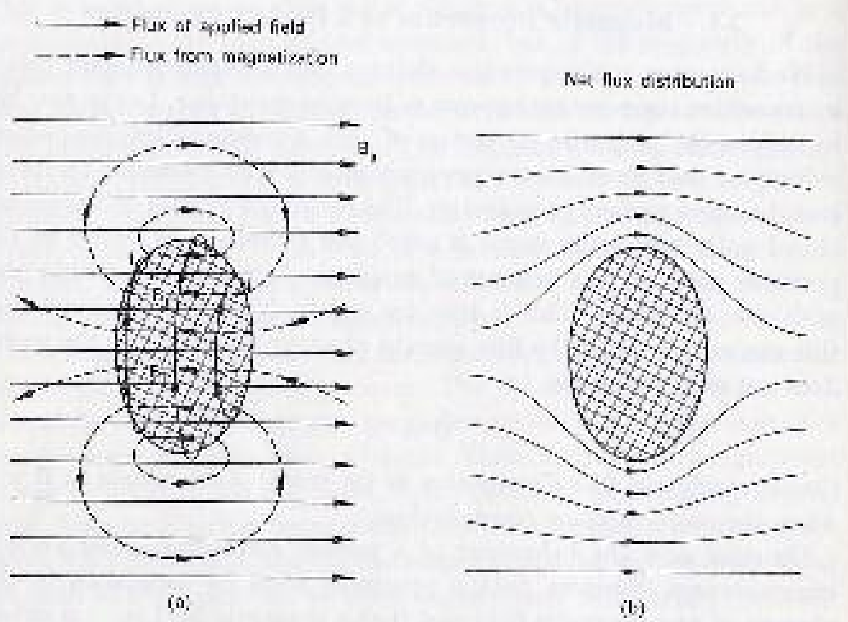


FIG. 7. Depiction of the surface currents that develop on a diamagnetic sample. (Adapted from Rose-Innes (op cit, p. 18)).

[13] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 20)

The London theory has other drawbacks. It does not determine the surface tension between the superconducting and normal regions within the same material. It does not explain the destruction of superconductivity by an applied critical magnetic field or by certain strengths of electric currents. It also fails to explain why the critical magnetic fields are different for thin films and bulk samples made from the same material. The London theory is a classical theory, where electrons are treated as classical particles rather than quantum mechanical wave functions, and it makes two assumptions that are now known to be incorrect. [14] The first assumption is that the London penetration depth $\lambda_L$, defined as $\lambda_L=(m/\mu_0 n_s e^2)^{1/2}$, where m is the electron's mass, e is its charge, and $n_s$ is the density of superconducting electrons, is independent of the magnitude of the applied magnetic field. The second assumption is that the London penetration depth is independent of the physical size of the material. These limitations of the London theory motivated Vitaly Ginzburg and Lev Landau to formulate their own theory of superconductivity that could explain the perplexing ideas about superconductivity that existed at that time. Ginzburg and Landau were able to clarify most of the issues that were left unexplained in the London theory.

## 2.2 The Ginzburg-Landau (GL) theory

The main difference between the GL theory and the London theory is that the former uses quantum mechanics in order to predict the effects of an applied magnetic field and the latter is a purely classical theory. The GL theory also turns out to be a phenomenological one since Ginzburg and Landau made certain assumptions in

[14] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 100)

describing the transition from the normal state to the superconducting state when there is no applied magnetic field on the material.

Ginzburg and Landau described the number of superconducting electrons using an "effective wave function" $\Psi$. The density of superconducting electrons, $n_s$, is associated[15] with the square of the magnitude of this function, $|\Psi|^2$. The theory also describes, close to the critical temperature $T_c$, the variation of the free energy[16] density of the superconducting state, $f_s$, from that of the normal state, $f_n$ , as a Taylor series expansion in $|\Psi|^2$:

$$f_s = f_n + \alpha|\Psi|^2 + (\beta/2)\,|\Psi|^4 + \ldots \qquad (1)$$

where a value of $\alpha=\alpha_0(T - T_c)$ and a positive constant for $\beta$ is assumed in order to have a stable superconducting state. Fortunately, we do not have to worry about any of the terms beyond the quadratic term in $\Psi$ when the temperature is sufficiently close to the critical value since that yields a good enough approximation to the free energy density of the superconducting state. [17]

In order to account for the energy of the magnetic field, Ginzburg and Landau made use of a theorem[18], which states that the motion of a charged particle that is subjected to a Lorentz force, such as the one produced by a magnetic field **B**, $q\mathbf{v}\times\mathbf{B}$, can be completely accounted for by making the substitution $\mathbf{p}\rightarrow \mathbf{p}-q\mathbf{A}$ in the kinetic energy

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[15] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 101)
[16] The Gibbs free energy G is a thermodynamic function given by G=E–TS, where E is the internal energy, S is the entropy, and T is the temperature of the sample. It is a useful device in comparing the relative energies between different states of materials.
[17] D.R. Tilley and J. Tilley, *Superfluidity and Superconductivity*, 3rd ed. (IOP Publishing, Bristol, 1990), pp. 296-298.
[18] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 102)

term[19], where **A** is the vector potential, defined by **B**=∇×**A**. A full-blown quantum mechanical expression would not be complete without the use of quantum mechanical operators, such as the use of -i∇∇ instead of the momentum **p**. After making all of these substitutions, the expression for the free energy density of the superconducting state in proper quantum mechanical language becomes[20],

$$f = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}\left|\left(-i\hbar\nabla - 2e\mathbf{A}\right)\psi\right|^2 + \frac{B^2}{2\mu_0} - \frac{1}{2}\mu_0 \mathbf{H_0}^2 \qquad (2)$$

where $\mathbf{H_0}$ is the applied magnetic field, **B** is the magnetic induction, *m* represents the mass of an electron, and *e* represents the fundamental unit of charge. In order to get an expression for the total free energy of the superconducting material, equation (2) has to be integrated over the whole volume of the material. Once this is done and after minimizing this expression with respect to the effective wave function Ψ, its complex conjugate $\Psi^*$, and the vector potential ***A***, the famous Ginzburg-Landau equations are obtained[21]:

$$\frac{1}{2m}\left(-i\hbar\nabla - 2e\mathbf{A}\right)^2\psi + \alpha\psi + \beta|\psi|^2\psi = 0 \qquad (3)$$

$$\mathbf{J_e} = \frac{-ie\hbar}{m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \frac{4e^2}{m}\psi^*\psi\mathbf{A} \qquad (4)$$

After careful analysis of equations (3) and (4), we would notice that the former is similar to Schrödinger's equation for a particle of mass *m*, charge *e*, and an energy

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[19] The kinetic energy term, in its quantum mechanical version, is the last term of eq. A4 of Appendix A. In the quantum mechanical version, the substitution is performed on the del operator rather than on the momentum **p** as can be seen in the appendix.

[20] Refer to Appendix A for a more detailed explanation of where the last few terms of equation 2 come from.

[21] Refer to Appendix A for a more detailed derivation of the Ginzburg-Landau equations.

eigenvalue -$\alpha$ except for a repulsive potential from the nonlinear term in $\psi$. Equation (4) is similar in form to the expression for the current density in quantum mechanics.[22] These equations accurately describe the dynamics of the charged particles in the superconducting material and, as we have just seen, they are expressed in quantum mechanical language. This means that whatever happens inside superconductors is in a quantum mechanical realm governed by quantum mechanical rules.

One of the many useful aspects of the Ginzburg-Landau theory is that its equations can be used to obtain solutions of the effective wave function $\Psi(x, y, z)$ and the vector potential $\mathbf{A}(x, y, z)$. These solutions to the GL equations minimize the free energy of the superconductor. It turns out that for weak applied magnetic fields, the GL equations reduce to the equations from the London theory, so it is a weak field approximation.[23] On the other hand, for high applied magnetic fields, the GL equations are only soluble through numerical means.

The GL theory was able to make some connections between the applied magnetic field and the penetration depth that had not been noticed before. For example, the GL theory predicts that $|\Psi|^2$ is a constant in the interior of an infinitely thick superconducting plate. Moreover, the density of superconducting electrons is inversely proportional to the magnitude of the applied magnetic field.[24] As was intimated earlier, the London penetration depth depends on the number of superconducting electrons at the surface, so we now a have a connection between the magnitude of the applied magnetic field and the penetration depth of the material. Ginzburg and Landau were also able to make a connection between the penetration depth and the thickness of superconducting thin films.

[22] The Royal Swedish Academy of Sciences (op cit, p. 8)
[23] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 102)
[24] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 103)

After an application of boundary conditions, the variation of the effective wave function will be dependent upon the thickness of the film. As we just saw, the penetration depth depends on the density of superconducting electrons, which is just the square of the magnitude of the effective wave function. We now have a penetration depth that depends on the thickness of the film.[25]

The GL theory also predicts the manner in which materials make their transitions from their superconducting states to their normal states. For thin films, the theory predicts that there will be second-order, or smooth, transitions as the applied magnetic field reaches the enhanced critical magnetic field value from below. For bulk superconductors, the behavior would be different. They would have a second-order transition in the absence of an applied magnetic field, but a first-order discontinuous transition in the presence of a magnetic field (Fig. 8). The GL theory also states the specific critical thickness values at which the transitions change from being first order to second order. The material will have a first order transition for $a > (5^{1/2}/2)\ \lambda$ and a second order transition for $a < (5^{1/2}/2)\ \lambda$.

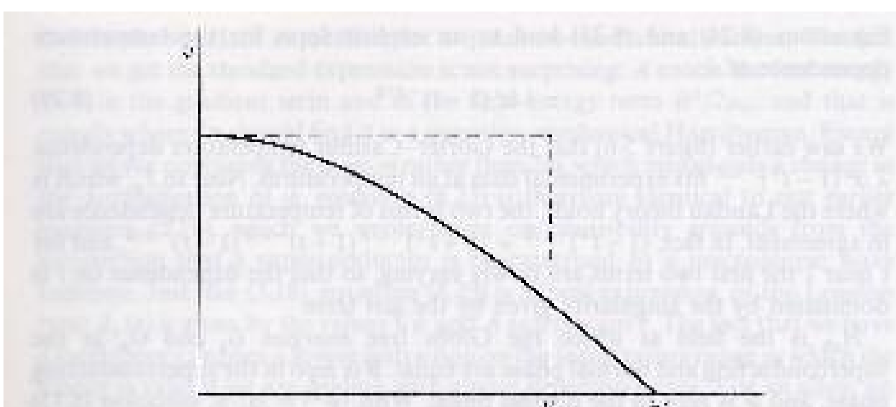

**Figure 8.** The broken lines correspond to a first order transition for bulk type-1 superconductors and the full curve corresponds to a second order transition for a thin-film type-1 superconductor. (Adapted from Tilley (op cit, p. 304)).

[25] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 103)

For the cases in which the thickness of the superconductors does not fit with the limits described above, the values of the critical magnetic field have to be obtained numerically. In these cases and when the values of the penetration depth are measured from bulk superconductors, the GL theory, like the London theory, only gives approximate values for the critical magnetic fields. However, a major advantage of the GL theory is that it correctly predicts the behavior of the transitions for superconducting materials as a function of their thickness.[26]

## 3. THE THEORY OF TYPE-II SUPERCONDUCTORS

As indicated above, there are two kinds of superconductors: type-I and type-II superconductors. It was originally thought that the characteristics outlined above applied to any kind of metal that became a superconductor at the critical temperature. Scientists eventually discovered some anomalies in certain samples. These included the presence of magnetic fields inside the superconducting regions of the sample as well as the appearance of normal regions within the superconductor. These anomalies were usually thought to be the effects of impurities within the sample. Alexei Abrikosov was the one who came up with the conclusion that these features, originally thought to be anomalies, were actually characteristics of a different type of superconductor, now known as a type-II superconductor. The main feature that distinguishes type-I from type-II superconductors is the value of the surface energy.

[26] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 104)

## 3.1 The concept of the surface energy

A surface energy will show up in a sample whenever there are both normal and superconducting regions within the same sample. This tends to happen at the critical magnetic field value, $H_c$, where normal samples become superconducting and vice versa.[27] When there are both normal and superconducting regions in the same material, the change in going from one region to the other is not a drastic one. As was discussed earlier, the magnetic fields that correspond to normal regions tend to penetrate into the superconducting regions up to a distance about equal to the penetration depth $\lambda$ and, over the same region, the density of superconducting electrons will increase to its highest value over a distance about equal to the coherence length $\xi$. In order to get stable normal and superconducting regions in the material, the free energy density in each of these regions must have the same value.[28]

As opposed to normal regions, the free energy density of superconducting regions will change due to two main factors. The first is that the free energy density will decrease due to the presence of superconducting electrons, which can be thought of as being more ordered than regular electrons. In other words, the higher the superconducting electron density, the lower the free energy density of that superconducting region. The free energy density is lowered by the amount $g_n$-$g_s$, where $g_n$ is the Gibbs free energy density of the normal phase and $g_s$ is the Gibbs free energy density of the superconducting phase.

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[27] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 80)
[28] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 81)

The second factor that will change the free energy density of the superconducting region is its magnetization. The purpose of this magnetization is to cancel the applied magnetic flux in order for the magnetic fields in its interior to be zero. An effect of this magnetization is to increase the Gibbs free energy density equal to $(1/2)\mu_0 H_c^2$.

The superconducting region will be in equilibrium if these two contributions to the free energy density exactly cancel out.[29] That is, if $g_n - g_s = (1/2)\mu_0 H_c^2$. This happens throughout most of the superconducting regions of the material except near the boundaries between the normal and superconducting regions, where the degree of order increases, and the free energy density decreases, over a distance about equal to the coherence length $\xi$ and the free energy increases over a distance about equal to the penetration depth $\lambda$ due to its magnetization. A difference in the $\xi$ and $\lambda$ values will lead to a departure from a state in equilibrium. The coherence length and penetration depth values depend on the material and do not always have the same value, which means that the specific contributions to the free energy density from the superconducting electrons and from the magnetization will not cancel out.[30] If the coherence length is longer than the penetration depth, the total free energy will increase, or have a positive value. The material is then said to have positive surface energy since this effect happens at the boundary, or surface, between the normal and superconducting regions (refer back to Fig. 5). Type-I superconductors have a positive surface energy. On the other hand, if the penetration depth is longer than the coherence length, the opposite effect will happen and the material will have a negative surface energy, which is an intrinsic characteristic of type-II superconductors.

[29] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 81)
[30] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 81)

## 3.2 The mixed state

The presence of positive and negative surface energy implies that, for type-I superconductors, if normal regions were to appear, these would only tend to increase the free energy of the material. In other words, this situation would be energetically unfavorable. This is why type-I superconductors always remain totally superconducting at values of an applied magnetic field smaller than the critical magnetic field value. On the other hand, type-II superconductors have a negative surface energy, which implies that the appearance of normal regions within the superconductor would tend to reduce the material's free energy and it would then be energetically favorable for them if many normal regions appear within the superconductor. These normal regions would be oriented parallel to the applied magnetic field and, in order to maximize the surface area to a certain volume of material so that the free energy of the metal is as low as possible, the shape of these normal regions, also called normal cores, is cylindrical in shape. The radius of these normal cores is also as small as possible in order to maximize the surface area. The periodicity of these normal cores is less than $10^{-7}$ m.[31] This condition is known as the mixed state and, since this only happens with materials that have a negative surface energy, it is an intrinsic property of type-II superconductors above the lower critical magnetic field $H_{c1}$.

As was shown earlier, materials that have a coherence length that is longer than its penetration depth will have a positive surface energy. The values of the coherence length and penetration depth depend on the material, but certain alloys and impure metals tend to have a longer penetration depth than the coherence length. This is due to the

[31] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 178)

shorter electron mean free path for these materials in both the normal and superconducting states.[32] This makes sense if we think of superconductors as having both normal electrons near the boundary between the normal and superconducting regions and superconducting electrons in the rest of the material. If the normal electrons have a shorter mean free path than the superconducting electrons, then it follows that the coherence length will be shorter. From Fig. 9, it is apparent that if the coherence length is shorter than the penetration depth, the free energy contributions from the magnetization and the electron ordering would not cancel near the boundary and the net free energy density would in fact be negative. This is why alloys and impure metals tend to be type-II superconductors.

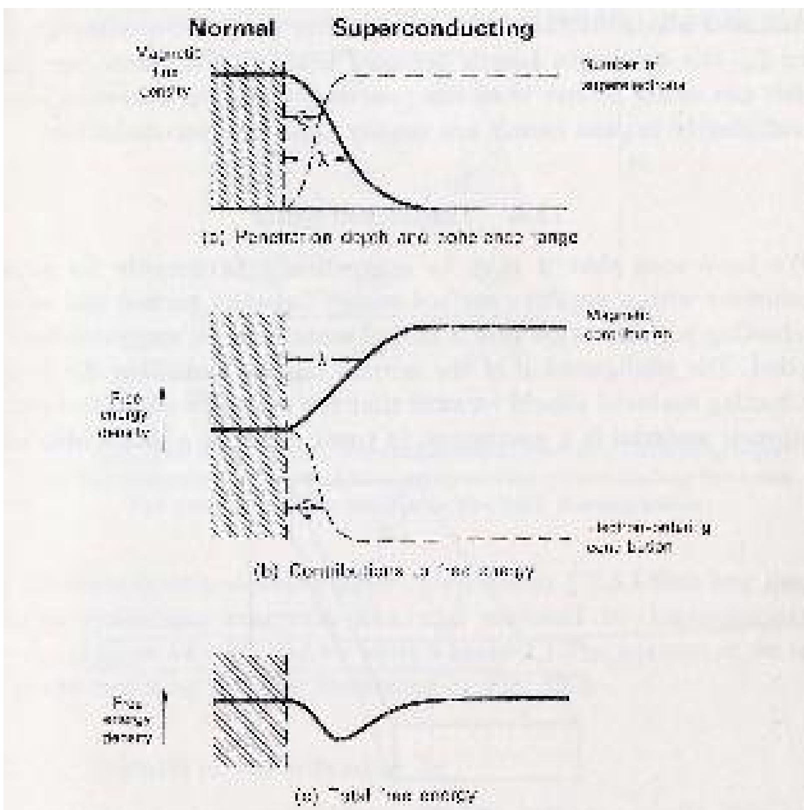


FIG. 9 . Plots depicting the emergence of negative surface energy. (Adapted from Rose-Innes (op cit. p. 175)).

[32] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 79)

Fig. 10 shows a magnetic field $H_a$ being applied to a type-II superconductor. Attentive readers will notice something odd about it. It shows a type-II superconductor with an applied magnetic field pointing in the upwards direction. The diamagnetic properties of superconductors will cause surface currents to circulate in the clockwise direction in order to produce a magnetic field that will cancel the applied magnetic field inside the superconductor. However, currents are seen to be circulating in the opposite direction along the surface of the normal cores. The reason for this phenomenon is related to a property of superconductors that have normal regions inside the material. Currents will develop along the surface of these normal regions in order to produce a magnetic flux that is parallel to the direction of the applied magnetic field. In other words, these normal regions are trying to oppose the canceling effects of the magnetic field inside the material caused by its superconducting regions.[33]

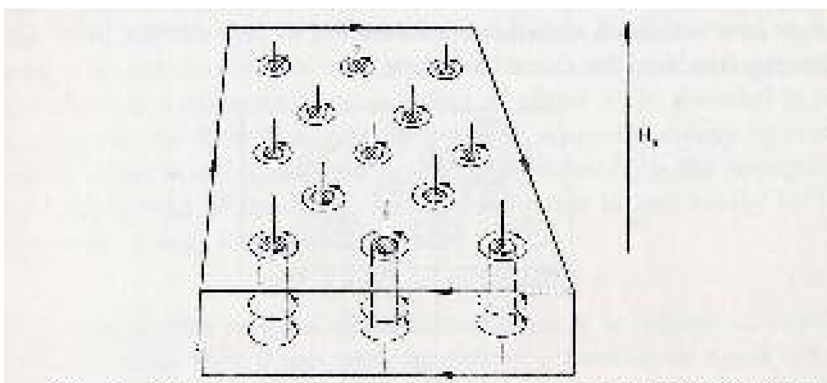

FIG. 10 The arrows encircling the superconductor represent the direction of the diamagnetic surface currents of the material. Inside the sample, there are normal cores with encircling supercurrent vortices producing a vertical magnetic flux. (Adapted from Rose-Innes (op cit, p. 177)).

Due to the small radius of the normal cores and the fact that the changes in the magnitude of the magnetic field and the density of superconducting electrons are gradual, the boundaries of these normal regions are hard to define. However, there is a periodicity

[33] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 27)

of the properties of these normal cores throughout the type-II superconductor.[34] That is, for every normal core within the superconductor, the density of the superconducting electrons will drop to zero at the center. This means that the diameter of a normal core can be approximated as two coherence lengths wide and that there is a very thin line of normal material along the center of each normal core. The magnetic flux density along the center of each core also decreases over a distance about equal to the penetration depth going towards the superconducting region. These periodic properties of the normal cores can be seen in Fig. 11.

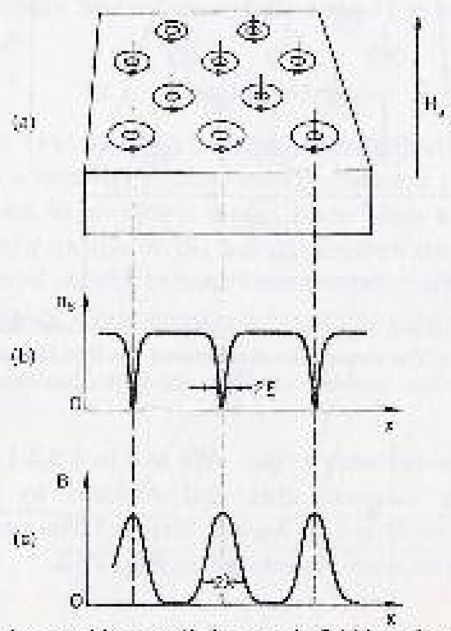


FIG. 11. The mixed state with an applied magnetic field just above the lower critical magnetic field value $H_{c1}$. (a) Lattice of normal cores. (b) The change in density of superconducting electrons with respect to position inside the superconductor. (c) Corresponding change in the flux density. (Adapted from Rose-Innes (op cit, p. 178)).

Since the radius of the normal cylinders can be approximated as being equal to the coherence length, some other approximations can be made regarding the value of the free energy increase or decrease at the normal cores. Since the number of superconducting

[34] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 178)

electrons decreases as we get near the normal core, the corresponding free energy will tend to increase due to a decrease in the order provided by the superconducting electrons. This increase in the free energy per unit length of core is given by $\pi\xi^2\,(1/2)\mu_0 H_c^2$. On the other hand, there will be a decrease in the free energy per unit length of core given by $\pi\lambda^2\,(1/2)\mu_0 H_a^2$, where $H_a$ is the magnitude of the applied magnetic field, due to fact that the normal regions are not diamagnetic.[35] If the normal regions were diamagnetic, there would have been an increase in the free energy. In order to have a net decrease in the free energy from normal core formation, the increase must be less than the decrease in the free energy per unit length of core:

$$\pi\xi^2\,(1/2)\mu_0 H_c^2 < \pi\lambda^2\,(1/2)\mu_0 H_a^2 \qquad (5)$$

For certain values of the critical magnetic field and applied magnetic field, it is apparent by looking at eq. 5 that the coherence length must be smaller than the penetration depth in order for normal cores to appear in the superconductor. Fortunately, this result agrees with fig. 9.

As stated earlier, it was Alexei Abrikosov who came up with a theoretical explanation for the various characteristics of type-II superconductors mentioned above. Abrikosov came to the conclusion that for superconductors with a Ginzburg-Landau parameter [36] value of $\kappa > 2^{-1/2}$, otherwise known as type-II superconductors, superconducting regions will start to appear at a certain value of an applied magnetic field which would correspond to a stronger magnetic field than the critical magnetic field value for type-I superconductors.[37] This magnetic field value is now known as the upper

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[35] A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 179)
[36] Recall that the Ginzburg-Landau parameter is given by $\kappa=\lambda/\xi$.
[37] The Royal Swedish Academy of Sciences (op cit, p. 9)

critical magnetic field $H_{c2}$, given by[38] $H_{c2}=H_c\kappa(2)^{1/2}$. Above this value, superconductivity would cease to exist. Abrikosov also predicted another value for a critical magnetic field, called the lower critical magnetic field $H_{c1}$, below which there would be no normal regions present in the superconductor. That is, there is a minimum magnetic field strength required for normal regions to appear in the material. This is intimated by eq. 5, which says that, for certain values of the coherence length and penetration depth, the applied magnetic field must be greater than a certain fraction of the critical magnetic field value. Abrikosov also concluded that a lattice of normal regions with periodically distributed magnetic fields, now known as an Abrikosov lattice, would be the most energetically favorable mechanism for a type-II superconductor[39] (Fig. 12).

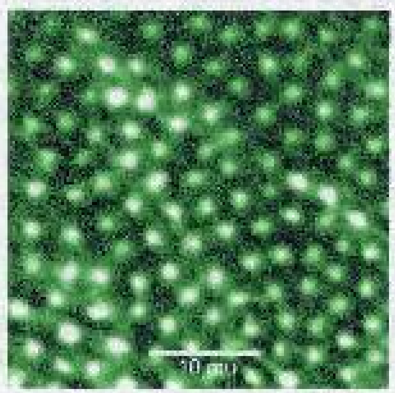

**Figure 12.** The Abrikosov lattice of magnetic flux lines in a sample of $NbSe_2$, which is a type II superconductor. This picture was taken using magneto optical imaging. (Adapted from the Royal Swedish Academy of Sciences (op cit, p. 10))

Abrikosov arrived at his normal core predictions through a careful analysis of the GL equations. More specifically, he noticed that for magnetic field values near the upper critical magnetic field, the effective wave function is very small and the nonlinear term in the first GL equation can be neglected. Abrikosov also noticed that his effective wave function solutions corresponded to vortices. This approach only works at high magnetic

[38] The Royal Swedish Academy of Sciences (op cit, p. 10)
[39] A.A. Abrikosov, *Fundamentals of the Theory of Metals*, (Elsevier Science Publishers, Amsterdam, 1988), pp. 414-415.

field values, but he was able to show that the vortex solutions also exist for weaker fields.[40]

After careful analysis of the first GL equation[41] (eq. 3), the reader will notice that it has a term $A-(\nabla/2e)\nabla\phi$. In order for the magnetic field to be a constant inside the superconductor, we must choose a suitable vector potential. A linear increase of the vector potential will do the job. For example, we can say $H_z=dA_y/dx$, where $A_y= H_z\ x$. In other words, we let the vector potential increase linearly in the x-direction. A problem immediately arises with the term from the first GL equation if we let the vector potential increase this way; it goes to infinity. This also implies that the free energy density goes to infinity. In order to keep the term finite, the increase in the vector potential must be compensated by a corresponding increase in the second part of the term, which corresponds to jumps in the phase. If we set the gradient of the phase equal to the change in the vector potential over one period of the lattice structure and solve for the period of the structure, the solutions for square and triangular lattices will be obtained.[42] The effective wave function will vanish at the vortices and their phase will change by $2\pi$ along a closed path around these vortices. [43]Abrikosov discovered these solutions in 1954, but did not publish them until 1957.

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[40] Abrikosov (op cit, p. 416)

[41] Recall that the order parameter is complex: $\Psi=|\Psi|e^{i\varphi}$. If we substitute this into eq. 3 and let the del operator work on the effective wave function, we get the term $\mathbf{A}-(\nabla/2e)\nabla\phi$ in the equation.

[42] Abrikosov (op cit, p. 417-418)

[43] Abrikosov (op cit, p. 416)

# 4. SUPERFLUID $^3$He – AN ANISOTROPIC SUPERFLUID

Up to this point, we have been considering theories of superconductivity, which occurs due to the formation of Cooper pairs of charged particles. We will now turn to the counterpart of superconductivity: superfluidity. The main mechanism behind this phenomenon is the formation of Cooper pairs of atoms rather than electrons. This means that the order parameter now describes the probability amplitude of the superfluid Cooper pairs. Cooper pair formation is partly responsible for the lack of viscosity in He liquids near the critical temperature for superconductors (around 3 or 4 K).

The reason why $^3$He and $^4$He have very different properties at low temperatures is due to the net spin of these atoms. From the upper indices of these atoms, we can see that one of them has an odd and the other an even number of nucleons. The former will have a nonzero net spin (I=1/2) and the latter will have a zero net spin since the individual spins cancel out. This means that the former will obey Fermi-Dirac statistics and the latter will obey Bose-Einstein statistics.

As was mentioned earlier, superconductivity involves Cooper pairs of electrons. Interestingly, these Cooper pairs are formed through electron-phonon interactions, mediated by the metallic lattice of atoms. The electrons in a Cooper pair have opposite spin projections and opposite momenta, making them structureless and bosonic objects[44]. As soon as these Cooper pairs are formed, they tend to remain in their lowest energy state. Since these are bosonic objects, they also tend to occupy the same space and this leads to

[44] D. Vollhardt and P. Wölfle (op cit, p. 4)

a higher degree of order in the material. Cooper pairs are also long lived due to a nearly full Fermi sea[45] near absolute zero (Fig. 13).

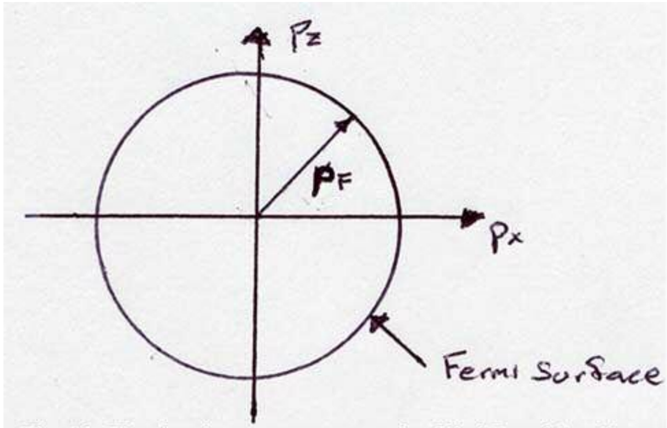


Fig. 13. The circular area represents the filled Fermi "sea". The radius $\mathbf{p}_F$ is known as the Fermi momentum. The momenta of conduction electrons in a normal metal are distributed uniformly throughout the Fermi sea.

This Cooper pairing mechanism cannot work for Helium-3 since it does not have an underlying crystal lattice that could mediate electron-phonon interactions. The reason why $^3$He atoms form Cooper pairs has to do with an intrinsic property of the liquid itself. A couple of the main features of the interatomic potential of $^3$He are the huge repulsive region at very short distances and the weak attractive van der Waals region at medium to long distances.[46] Due to the huge repulsive core, the variation in the potential does not change much between the relative angular momentum values of $l$=0 and $l$=1. However, the $^3$He atoms always settle into the $l$=1 state since the oblong shape of the electron cloud of each $^3$He atom due to the relative angular momentum of $l$=1 allows the atoms to be more closely packed together than the spherical shape of the electron cloud due to the relative angular momentum of $l$=0.

[45] This concept refers to the three-dimensional momentum space, represented as a sphere of radius $p_F$ of the constituent electrons of the metal. The sea is said to be full when all the quantum states with a kinetic energy less than that which corresponds to a momentum $p_F$, known as the Fermi energy, are taken. This happens at absolute zero. (A.C. Rose-Innes and E.H. Rhoderick (op cit, p. 121))
[46] D. Vollhardt and P. Wölfle (op cit, p. 5)

Superfluids have different phases like matter's gas, liquid, and solid phases. The superfluid phases are classified into three main categories: A, B, and $A_1$ phases. As we shall see shortly, these phases have very different properties, but they all have two things in common: the constituent Cooper pairs of atoms have a net spin S=1 and orbital angular momentum *l*=1 as opposed to Cooper pairs of electrons in superconductors, with S and *l* both equal to zero.[47] These characteristics lead to the superfluid Cooper pairs being known as having a "spin-triplet p-wave" pairing while superconducting Cooper pairs are in the "spin-singlet s-wave" state. The properties of $^3$He Cooper pairs are much more complex than those of superconducting Cooper pairs since the former has 3 spin and 3 orbital angular momentum substates, producing a total of nine substates for the Cooper pair wave function. On the other hand, superconducting electron Cooper pairs only have a single state.

The superfluid A-phase, also known as the axial p-wave state, has many interesting characteristics. The spins of the individual atoms that form the Cooper pair are perpendicular to the axis of orbital motion.[48] The energy gap[49] function also has the property that it has zero points along the direction of the orbital angular momentum projection on the Fermi surface[50] (Fig. 14). There are only two spin states allowed in the A-phase: S=±1. This phase can be described by a state known as the ABM state, appropriately named after its discoverers: P.W. Anderson, W. Brinkman, and P. Morel. The A-phase and the ABM state are thus analogous to each other.

[47] D. Vollhardt and P. Wölfle (op cit, p. 8)
[48] O.V. Lounasmaa and G.R. Pickett: *The $^3$He Superfluids*, Scientific American, June 1990, p. 105.
[49] The energy gap is an energy range around the Fermi energy that is forbidden for electrons. This energy comes about from the electron-phonon interactions. (W. Buckel, *Superconductivity: Fundamentals and Applications*, (VCH Publishers, New York, 1991), pp. 39-40)
[50] The Fermi surface is a surface, at a certain radius, of the Fermi sea in momentum space. (Abrikosov (op cit, p. 25))

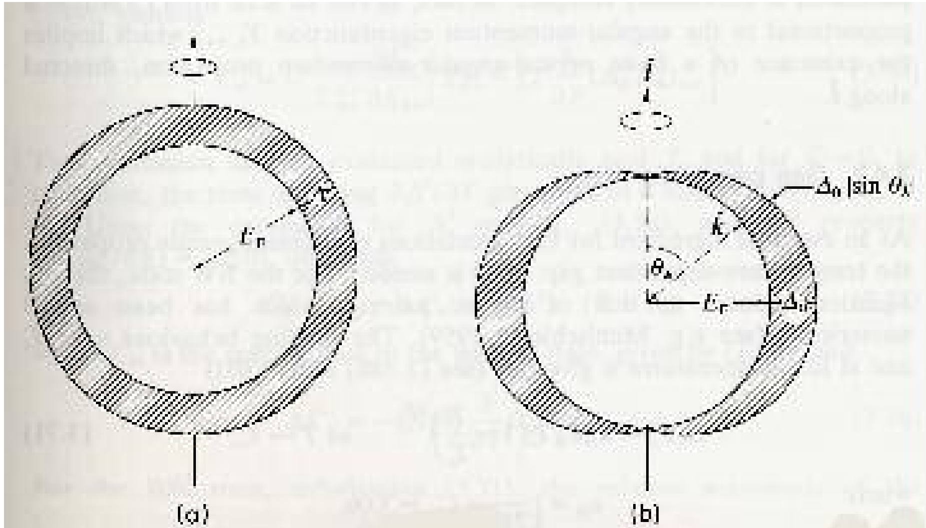


Figure 11 (a) The "(pseudo) isotropic" or BW state with its associated energy gap represented by the shaded area. (b) The "axial" or ABM state with associated energy gap that disappears in two places along the direction of angular momentum. Fermi energy and energy gap not drawn to scale. $E_F$ refers to the Fermi energy. $\Delta$ is the average energy gap, $\Delta_0$ is the maximum value of the energy gap, and k-hat is the unit vector in momentum space.
(Adapted from Vollhardt (op cit. p. 81)).

As opposed to the A-phase, in the superfluid B-phase all three spin substates are allowed: (S=±1 and S=0), and it occurs at lower temperatures than the A-phase. Its energy gap function is isotropic even though the Cooper pair wave function is intrinsically anisotropic. This phase can be described by a state known as the BW state, named after R. Balian and N.R. Werthamer. Another phase known as the $A_1$ phase occurs at the upper temperature limit of the A-phase, but at much higher applied magnetic field values. The only allowed spin states for the $A_1$ phase are S=±1 states. The $A_2$ and $B_2$ phases are basically the same as the A and B phases except with slightly different structures due to the presence of an applied magnetic field (Table 1).[51] The P-T-H phase diagram, where the H refers to an applied magnetic field value, shown in Fig. 15 should clarify the characteristics of the various superfluid phases.

[51] D. Vollhardt and P. Wölfle (op cit, p. 7)

Table 1 Outline of the phases of superfluid $^3He$. (Adapted from Vollhardt (op cit, p. 8)).

| External magnetic field $H$ | Stable phases |
|---|---|
| $H = 0$ | A phase<br>B phase |
| $H \neq 0$ | $A_1$ phase<br>$A_2$ phase ($\equiv$A phase in a magnetic field)<br>$B_2$ phase ($\equiv$B phase in a magnetic field) |

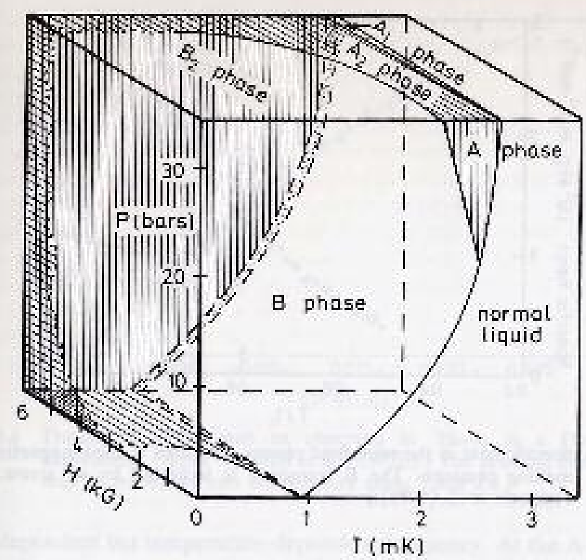


Figure .5 P-T-H phase diagram for superfluid $^3He$ showing the superfluid phases A, $A_1$, $A_2$, B, and $B_2$ as well as a normal Fermi liquid. The $A_2$ and $B_2$ phases are the A and B phases with an applied magnetic field. (Adapted from Vollhardt (op cit. p. 99)).

Scientists eventually found a way of studying the properties of the superfluid phases that were just mentioned and identifying the order parameter[52] structure of these phases. The technique that they use is called nuclear magnetic resonance (NMR). In this technique, the object being studied would be subjected to a strong magnetic field $H_0$ in a certain direction, which shall be called the z-direction. The spin of the objects being studied will then precess about the z-direction. A transition in the z component of the spin from a positive to a negative value can be induced if another, weaker, magnetic field is

[52] Recall that the order parameter refers to the probability amplitude of the superfluid Cooper pairs.

applied in a direction perpendicular to the $H_0$ field. A strong "resonance" occurs at a particular frequency, known as the Larmor frequency, which is given by $\omega_L = \gamma H_0$, where $\gamma$ is the gyromagnetic ratio of the nucleus. For interactions that conserve the spin, the resonance would still occur at the Larmor frequency. However, for interactions that do not conserve spin, such as the spin-orbit interaction due to the nuclear spins' dipole coupling of the Cooper pair members, a small shift in the resonant frequency was expected. What actually happened was a very big shift in the resonant frequency.[53] Anthony Leggett attributed this phenomenon to "spontaneously broken spin-orbit symmetry" of the Cooper pairs, where the spin and orbital angular momentum directions are characterized by some macroscopic order, such as the configurations depicted in Fig. 1(d) and (e).[54] He was able to come up with an expression for the expected shift in the resonant frequency when a superfluid sample in the A-phase is being used:

$$\omega_t^2 = \omega_L^2 + \Omega_A^2(T) \tag{6}$$

where $\omega_t$ is the value of the new transverse resonant frequency and $\Omega_A$ is a temperature-dependent quantity proportional to the dipole coupling constant.[55] Aside from a transverse resonant frequency, Leggett predicted a longitudinal resonant frequency for both the A and B phases, where a high-frequency magnetic field would now be directed

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[53] D. Vollhardt and P. Wölfle (op cit, p. 18)
[54] D. Vollhardt and P. Wölfle (op cit, p. 19)
[55] The coupling constant between the nuclear spins of the $^3He$ atoms, which reflects the fact that there is a weak attraction between the atoms due to dipole-dipole interactions, is given by

$$g_D \; [illegible] \frac{[illegible]^2}{a^3} \; [illegible] \frac{[illegible]}{E_F} \; [illegible]\, n$$

where $\mu_0$ is the nuclear magnetic moment, $a$ is the average atomic distance, $\Delta(T)$ is the average energy gap, $E_F$ is the Fermi energy, and $n$ is the particle density. The first factor corresponds to the average dipole energy of the atoms at a distance $a$ and the second factor is a measurement of the probability for the two atoms to form a Cooper pair. (D. Vollhardt and P. Wölfle (op cit, p. 13)) along the direction of the constant magnetic field $H_0$. He predicted the following value for the A-phase:

$$\omega_l=\Omega_A(T) \tag{7}$$

where the term on the right is the same as the one in eq. 6.

Leggett was the first to identify the A phase with the ABM state, but was buffaloed by the fact that, according to "weak coupling theory", the BW state is the one that is always in the lowest energy state. That is, superfluids would naturally prefer to be in the lowest energy state, so Leggett did not know why the A phase should exist at all. In order to understand this, effects from "strong coupling theory" need to be taken into account.

P.W. Anderson and W. Brinkman were able to explain why the A phase occurs where it does in the P-T-H diagram instead of the BW state. There is an effect that helps explain why the A phase occurs at high pressures: "[it] is based on a feedback mechanism: the pair correlations in the condensed state change the pairing interaction between the $^3He$ quasiparticles[56], the modification depending on the actual state itself."[57] An example that Anderson and Brinkman considered was the role of spin fluctuations, which are more pronounced at higher pressures.

[56] Not all helium-3 atoms pair up into Cooper pairs. The empty places, which represent the places that free helium-3 atoms would otherwise occupy (and pair up with another helium-3 atom to form a Cooper pair), are known as "holes". That is, there are unpaired atoms and corresponding "shadow" particles called "holes". The free atoms are known as quasiparticles and the holes are known as quasiholes. It turns out that at high momenta, the particlelike properties dominate while at low momenta, the holelike properties dominate. (O.V. Lounasmaa and G.R. Pickett (op cit, p. 108))
[57] D. Vollhardt and P. Wölfle (op cit, p. 12)

# 5. CONCLUSION

As has been intimated in our discussion of superconductors and superfluids, a reason why they are so important is that they allow us to see quantum mechanical behavior macroscopically, so these materials constitute a medium that physicists can use to directly analyze quantum theory. The 2001 Nobel Prize, awarded to Eric A. Cornell, Wolfgang Ketterle, and Carl E. Wieman "for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates", is further evidence of the importance of being able to see quantum mechanical effects macroscopically.

Besides the importance of superconductors and superfluids themelves, the theories behind them have allowed scientists to gain a deeper understanding of other phenomena. More specifically, the Ginzburg-Landau theory is relevant to many fields in physics. An example from particle physics is string theory. Abrikosov's theory of type-II superconductors is important in the study of high-temperature superconductors, which are extreme examples of type-II superconductors. Leggett's theory also helps explain complex phase transitions in many fields of physics, including particle physics, cosmology, and liquid crystal physics.

# APPENDIX A: THE GL EQUATIONS[58]

The Landau theory of phase transitions regards these processes as going from an ordered state to a disordered state. An order parameter $\phi$ may be defined as

$$f = \frac{n_+ - n_-}{n_+ + n_-} \qquad \text{(A1)}$$

where $n_+$ is the number of spins pointing upwards and $n_-$ is the number of spins pointing downwards. We are only concerned with the temperature region near $T_c$, where the order parameter is small. Therefore, we can expand the free energy F as a power series in $\phi$:

$$F = F_n + \lambda\phi + \alpha\phi^2 + \gamma\phi^3 + (1/2)\beta\phi^4 \qquad \text{(A2)}$$

where the coefficients are functions of T and can be expanded in powers of ($T - T_c$). For $T > T_c$, the order parameter will have a minimum at $\phi = 0$, where we can assume the same number of upward spins and downward spins. If we think of $\lambda$ as a derivative of $F$ with respect to the order parameter, we can see that it would be zero. In fact, it is zero for all T, if we think in terms of the Taylor expansion of $\lambda$. It also turns out that the term $\phi^3$ does not even occur in superconductors, so we now have a simpler version of eq. A2:

$$F = F_n + \alpha(T)\phi^2 + (1/2)\beta(T)\phi^4. \qquad \text{(A3)}$$

In order to apply the Landau theory to superconductors, we need to regard the wave function as the order parameter $\phi$. This means that eq. A3 must now be a function of position. After adapting the wave function to eq. A3, we will get an additional “kinetic energy” term proportional to $y^2$:

$$f \quad f_n + a \quad 2 \quad 2m \quad y \qquad \text{(A4)}$$

[58] Adapted from D.R. Tilley and J. Tilley (op cit, pp. 298-302)

where the last term's coefficient has that form due to convention in the GL theory and the square of the wave function still represents the superconducting electron density. Through dimensional analysis, we can see that a certain ratio of coefficients turns out to represent a length:

$$\xi = \left(\frac{\hbar^2}{2m|\alpha|}\right)^{1/2} \tag{A5}$$

which is the definition of the coherence length.

If **B** is the total induction in the superconductor and we take into account an applied magnetic field $\mathbf{H_0}$, we will expect the currents produced by the superconducting electrons to produce a magnetic field given by $\mathbf{B} - \mu_0\mathbf{H_0}$. Thus, we can write the following equation:

$$\nabla \times (\mathbf{B} - \mu_0 \mathbf{H_0}) = \mu_0 \mathbf{J_e} \tag{A6}$$

where $J_e$ is the current produced by the superconducting electrons.

Equation A4 now has to be modified to include the magnetic field effects by implementing the following substitution that takes into account the vector potential **A**:

$$\nabla \rightarrow \nabla - \frac{2ie}{\hbar}\mathbf{A}.$$

Implementing these changes and adding the magnetic field energy, we get:

$$f = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}\left|(-i\hbar\nabla - 2e\mathbf{A})\psi\right|^2 + \frac{(\mathbf{B} - \mu_0\mathbf{H_0})^2}{2\mu_0} - \frac{1}{2}\mu_0\mathbf{H_0}^2 \tag{A7}$$

where the last term represents the magnetic energy from the coils that produce the applied magnetic field $\mathbf{H_0}$. If we integrate eq. A7 over the volume of the material, we would get the Helmholtz free energy, which is given by F = U – TΣ. In this last expression, U is the internal energy of the superconductor in the presence of an applied magnetic field, and Σ is its entropy. At equilibrium, with a magnetization **M**, the superconductor satisfies the following thermodynamic relationship for the internal energy differential:

$$dU = T d\Sigma + \mathbf{H_0} \cdot \mathbf{M}. \tag{A8}$$

The Gibbs free energy, given in eq. A9, should now be minimized in order to find the stable state of the superconductor with an associated temperature value T and applied magnetic field value $\mathbf{H_0}$.

$$G(T, \mathbf{H_0}) = U - T\Sigma - \mathbf{H_0} \cdot \mathbf{M}. \tag{A9}$$

The Gibbs free energy can be obtained by integrating the Gibbs free energy density over the volume of the material:

$$G = \int g \, dV. \tag{A10}$$

The Gibbs free energy density is now given by the following function:

$$g = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}\left|\left(-i\hbar\nabla - 2e\mathbf{A}\right)\psi\right|^2 + \frac{1}{2\mu_0}\left|\nabla\times\mathbf{A}\right|^2 - \frac{1}{2}\mu_0 \mathbf{H_0}^2. \tag{A11}$$

A very useful mechanism in minimizing functions is the Euler-Lagrange equation of the calculus of variations. Since the Gibbs free energy depends on the wave function $\psi$ and the vector potential **A**, we can apply the Euler-Lagrange equations the following way:

$$\frac{\delta G}{\delta \psi^*} = 0 \tag{A12}$$

$$\frac{\partial g}{\partial \psi^*} - \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial g}{\partial(\partial_j \psi^*)}\right) = 0. \tag{A13}$$

Eq. A12 is just shorthand notation for eq. A13. Note that in eqs. A12 and A13, $g$ is being differentiated with respect to $\psi^*$. Recall that the wave function is complex, so $\psi^*$

represents the complex conjugate. After performing the operation depicted in eq. A13 and using the fact that div **A** = 0, eq. A13 becomes

$$\frac{1}{2m}\left(-i\nabla\nabla - 2e\mathbf{A}\right)^2\psi + \alpha\psi + \beta\left|\psi\right|^2\psi = 0 \ . \qquad \text{(A14)}$$

Eq. A14 is the first GL equation. Assuming we know the vector potential, we can use the first GL equation to obtain the behavior of the wave function throughout the superconductor.

Ginzburg and Landau also came up with some boundary conditions to complement the differential equation of A14. They did this by saying that the variation of G when $\psi^*$ varies by $\delta\psi^*$ includes, first of all, the volume integral of the left-hand side of eq. A13 multiplied by $\delta\psi^*$, and also the following surface integral:

$$I_s = \oint \delta\psi^* \left(\frac{\partial g}{\partial \nabla\psi^*}\right)\cdot \mathbf{n}\, dS \qquad \text{(A15)}$$

where **n** is a normal vector to the surface of the superconductor. Eq. A15 is usually neglected since $\psi^*$, and therefore $\delta\psi^*$, is usually taken to be zero at the surface boundaries. However, such a standing wave condition cannot be applied to superconductors since it would mean, for example, that the critical temperature would oscillate with the thickness of a thin film specimen and this is not the case. Ginzburg and Landau then argued that the second term in the integral must be zero:

$$\mathbf{n}\cdot\left(\frac{\partial g}{\partial \nabla\psi^*}\right) = 0 \ . \qquad \text{(A16)}$$

Carrying out the operation depicted in eq. A16 would yield:

$$\mathbf{n}\cdot(-i\nabla\nabla - 2e\mathbf{A})\psi = 0 \ . \qquad \text{(A17)}$$

Eq. A17 only holds at the boundaries between superconductors and insulators. As we have seen, when a superconducting regions is right next to a normal, metallic, region the wave function is able to penetrate a small distance into the normal region.

To get the second GL equation, we have to minimize the Gibbs free energy with respect to the vector potential, which implies the use of the Euler-Lagrange equation:

$$\frac{\partial g}{\partial A_i} - \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial g}{\partial\left(\frac{\partial A_i}{\partial x_j}\right)}\right) = 0 \quad \text{(A18)}$$

The second term in eq. A18 is equal to $(1/\mu_0)$ curl curl **A**, and from the definition of the vector potential, we can write it as $(1/\mu_0)$ curl **B**. This last expression is immediately recognized as part of one of Maxwell's equations, which is equal to the current density $\mathbf{J_e}$. After performing the rest of the necessary calculus on eq. A11, eq. A18 becomes

$$\mathbf{J_e} = \frac{-ie\hbar}{m}\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right) - \frac{4e^2}{m}\psi^*\psi\mathbf{A} \quad \text{(A19)}$$

which is the second GL equation. It is similar in form to the quantum mechanical description of a current. A change in the normalization of $\psi$ is required to get the expression in terms of the mass of the Cooper pairs, $2m_e$.

# Selective Annotated Bibliography

[1] A.A. Abrikosov, *Fundamentals of the Theory of Metals*, (Elsevier Science Publishers, Amsterdam, 1988).

This book takes an advanced and comprehensive approach to the theory of superconducting metals as well as normal metals. I would say this is at the same level as Tilley's book, but unfortunately, it uses archaic cgs units.

[2] W. Buckel, *Superconductivity: Fundamentals and Applications*, (VCH Publishers, New York, 1991).

This is another introductory book on superconductivity theory and is meant for non-specialists. It is good resource for those who are seeking a more descriptive approach to the theory. It has nice pictures too.

[3] G.W. Crabtree and D.R. Nelson: *Vortex Physics in High-Temperature Superconductors*, Physics Today, April 1997.

This is a useful resource for those who want an introduction to the dynamics of vortices in superconductors.

[4] A. J. Leggett: *A theoretical description of the new phases of liquid $^3$He,* Rev. Mod. Phys. **47**, 331 (1975).

This is Leggett's theoretical research paper on the subject of the phases of liquid $^3$He. Therefore, it includes the most elegant mathematics and would only recommend it to those interested in the kind of mathematics used and to advanced graduate physics students interested in superfluid phases.

[5] O.V. Lounasmaa and G.R. Pickett: *The $^3$He Superfluids*, Scientific American, June 1990.

This is a nice introduction to the phenomenon of superfluidity in helium-3 with lots of good pictures. I would recommend this articles as a starting point, before heading into introductory books on superfluidity.

[6] A.C. Rose-Innes and E.H. Rhoderick, *Introduction to Superconductivity*, (Pergamon Press, London, 1969).

As the name states, this is only an introduction to the subject and the mathematical formalism is not as rigorous as other, more advanced, texts. However, I found this book to be the most helpful since I found the more advanced texts to be too sophisticated.

[7] The Royal Swedish Academy of Sciences – The Information Department: *Superfluids and superconductors: quantum mechanics on a macroscopic scale*, http://www.nobel.se/physics/laureates/2003/phyadv03.pdf, posted on 7 October 2003, accessed on 14 October 2003.

This is a nice summary of the major contributions of the 2003 Nobel laureates as well as a general advanced introduction, in the advanced undergraduate level, to the phenomena of superconductivity and superfluidity.

[8] J.R. Schrieffer, *Theory of Superconductivity*, (W.A. Benjamin, New York, 1964).

This text presents a more rigorous approach to the study of superconductivity than the Rose-Innes book. This book is obviously at the graduate level, so I would recommend this to those who are curious about the mathematical formalisms behind superconductivity theory.

[9] D.R. Tilley and J. Tilley, *Superfluidity and Superconductivity*, 3rd ed. (IOP Publishing, Bristol, 1990).

I found this text to be easier to follow than both Vollhardt's and Schrieffer's books, but it is still more advanced than the Rose-Innes book. I would recommend this to those who already know the basics of superconductivity and superfluidity.

[10] D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3*, (Taylor and Francis, London, 1990).

This is an advanced, graduate-level text with a comprehensive examination of the superfluid phases of helium 3. I found the mathematics to be too elegant, but I found its introduction to be very useful and more understandable.