Maximum Arc Digraph with a Given Zero Forcing Number

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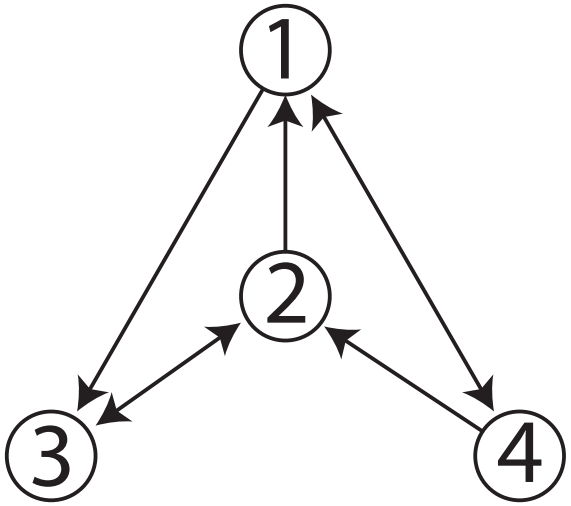
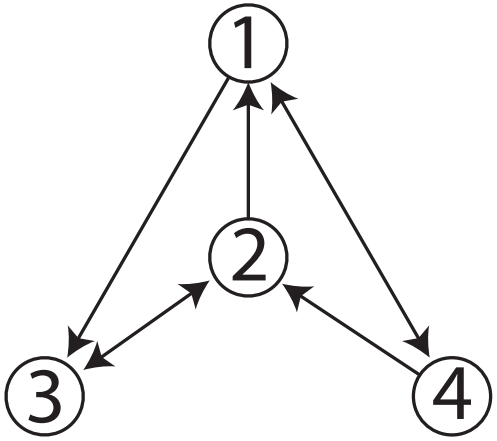
Introduction



An example of a digraph

A digraph $\Gamma = (V, E)$, is a vertex set, V, and an arc set of ordered pairs, E, where $(u, v) \in E(\Gamma)$ if $u, v \in V(\Gamma)$ and there exists an arc in Γ that points from u to v.

Matrices for Digraphs



$$\begin{bmatrix} ? & 0 & * & * \\ * & ? & * & 0 \\ 0 & * & ? & 0 \\ * & * & 0 & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 5 & 8 & 0 \\ 0 & 13 & 0 & 0 \\ 21 & 34 & 0 & 55 \end{bmatrix}$$

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For any digraph Γ, $mr(\Gamma) \neq n$ and $M(\Gamma) \neq 0$.

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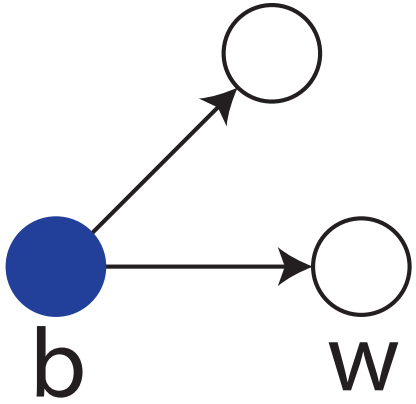

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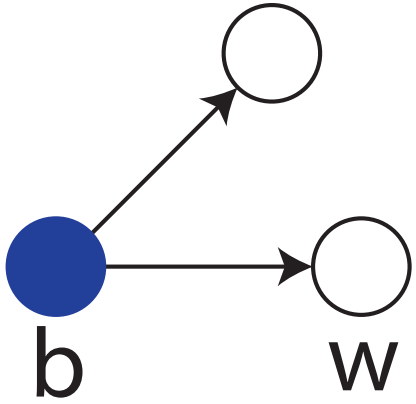

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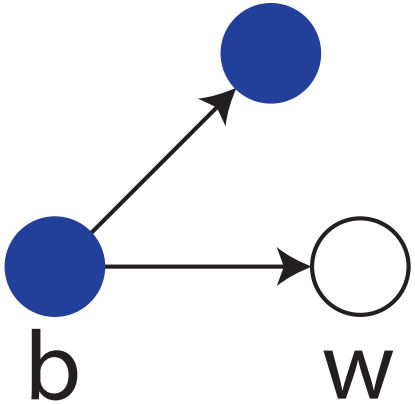

b can force w

The zero forcing number, $Z(\Gamma)$, is the minimum number of vertices that need to be colored blue in order to force the rest of the graph to be colored blue through the color change rule.

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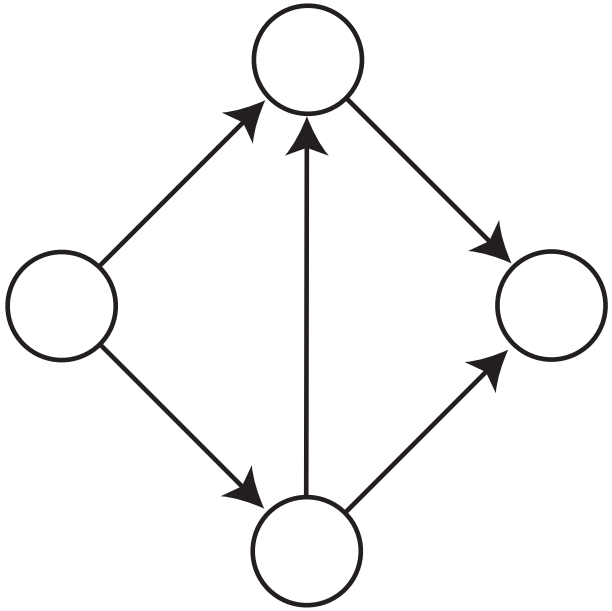

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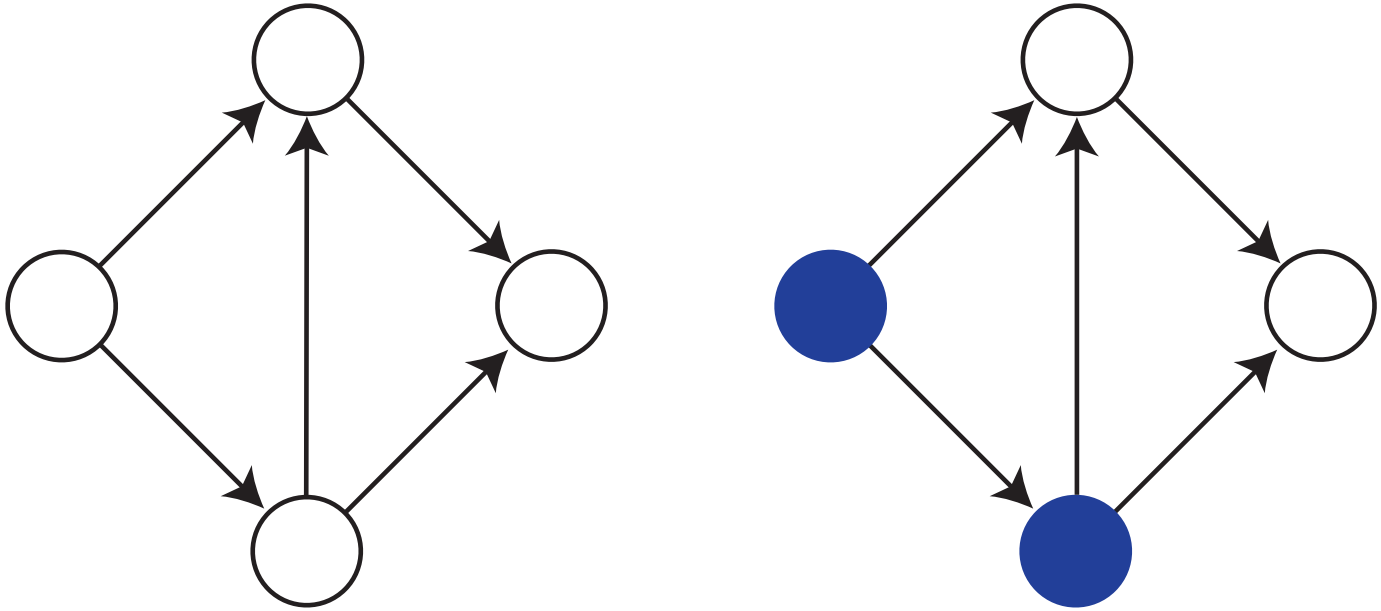

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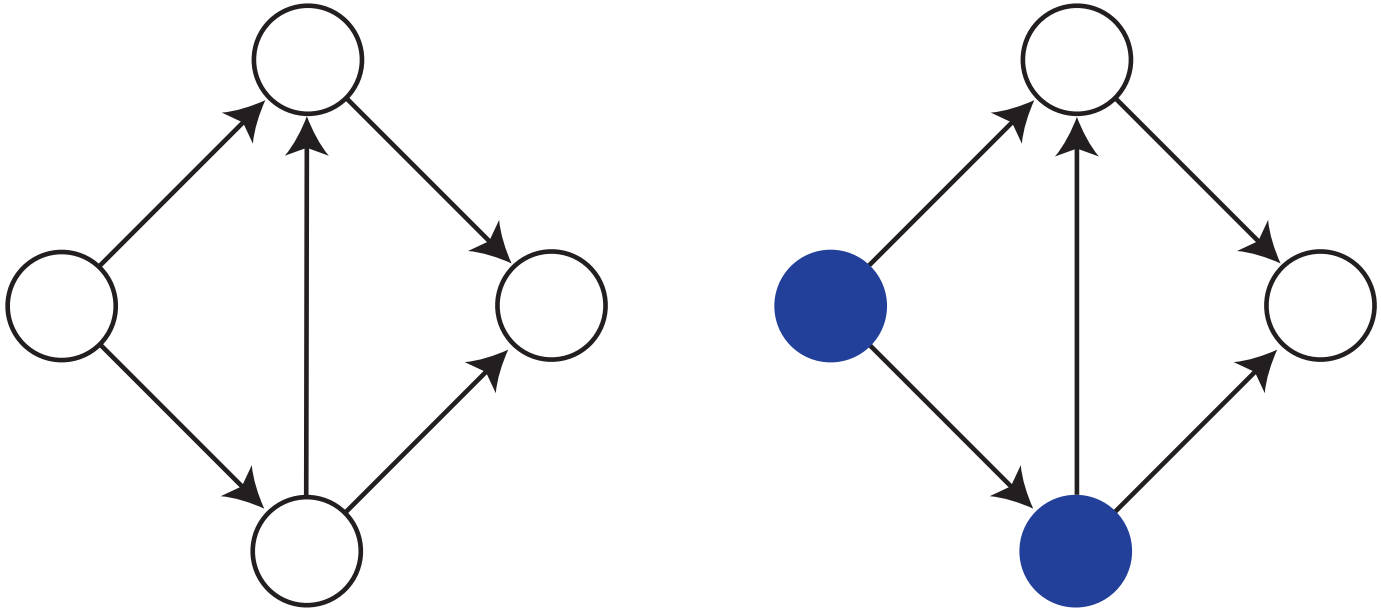
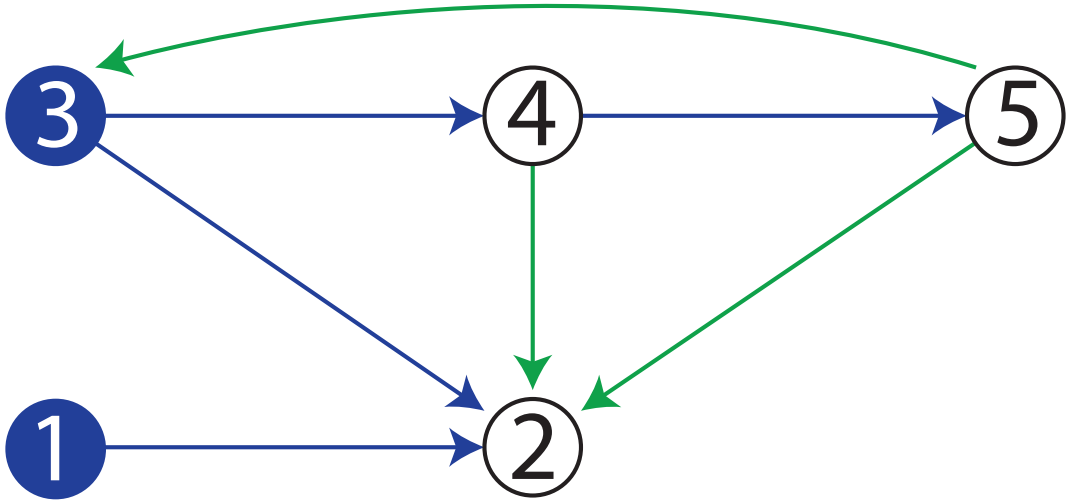


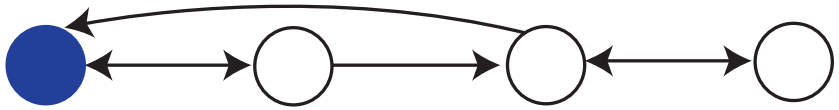
For any digraph Γ, $M(\Gamma) \leqslant Z(\Gamma)$ (Barioli et al., 2008)(Hogben, 2010).

Given a zero forcing set and a corresponding chronological list of forces, a backward arc is any arc $(u, v) \in E(\Gamma)$ such that v is forced before u. A forward arc is any arc that is not a backward arc.

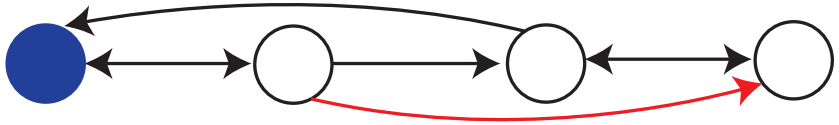


Hessenberg Paths

A path $(v_1, ..., v_k)$ in a digraph Γ is Hessenberg if it is a path that does not contain any arc of the form (v_i, v_j) with $j > i + 1$.



A Hessenberg Path



Adding an illegal arc

Theorem (Hogben, 2010)

$Z(\Gamma) = 1$ if and only if Γ is a Hessenberg path.

Path Cover

- A path cover of Γ is a set of vertex disjoint Hessenberg paths that includes all vertices of Γ.

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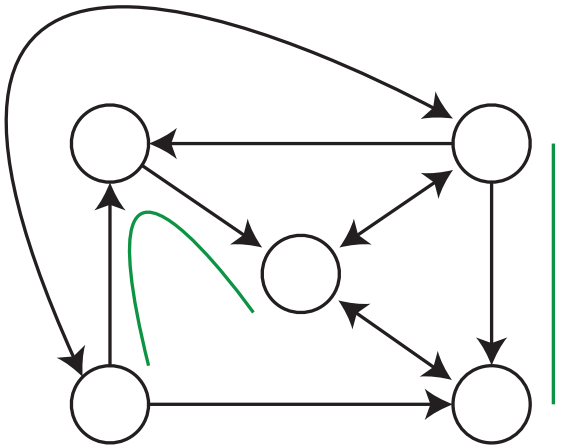

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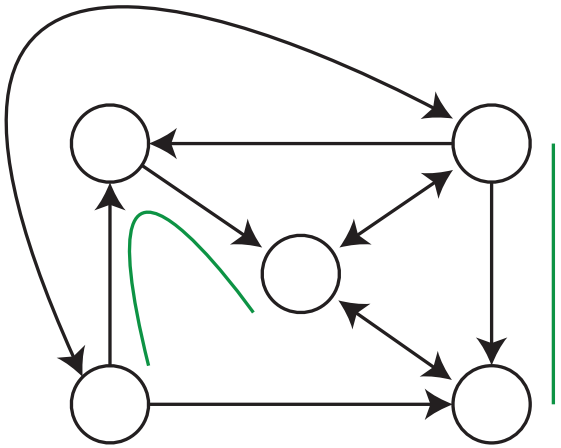

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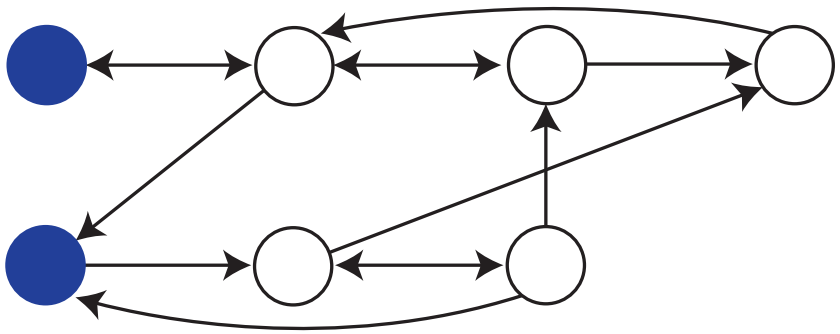
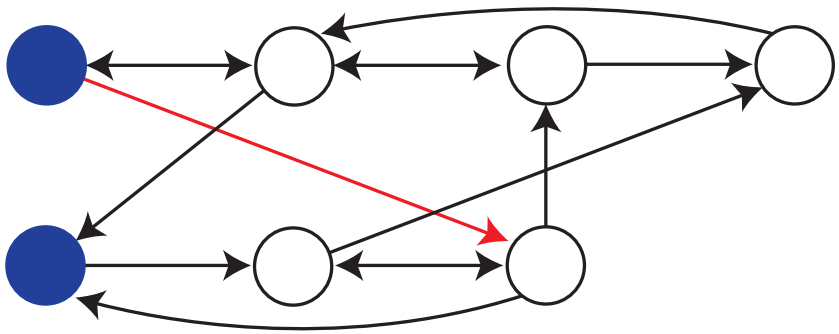
- For any digraph Γ, $P(\Gamma) \leqslant Z(\Gamma)$ (Hogben, 2010).

Digraph of two parallel Hessenberg paths



A Parallel Hessenberg Path



Adding an illegal arc

Important Theorems

Theorem (Berliner et al., Under Review)

$Z(\Gamma) = 2$ if and only if Γ is a digraph of two parallel Hessenberg paths.

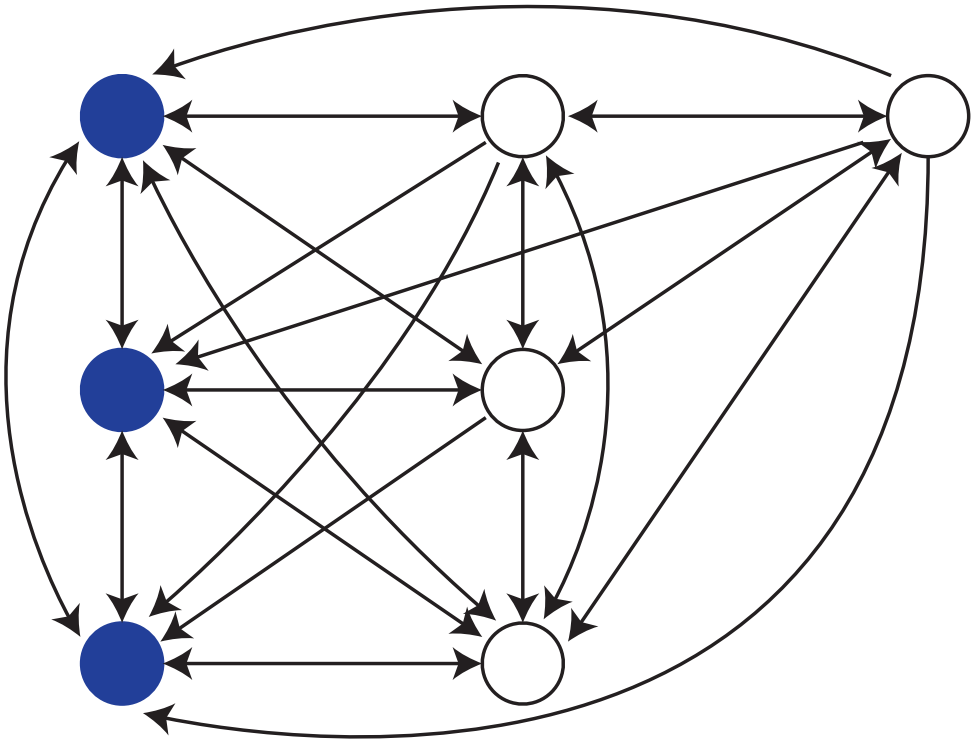
Theorem (Hogben, 2010)

Suppose Γ is a digraph and $\mathfrak{F}$ is a chronological list of forces of a zero forcing set B. A maximal forcing chain is a Hessenberg path.

Our Question

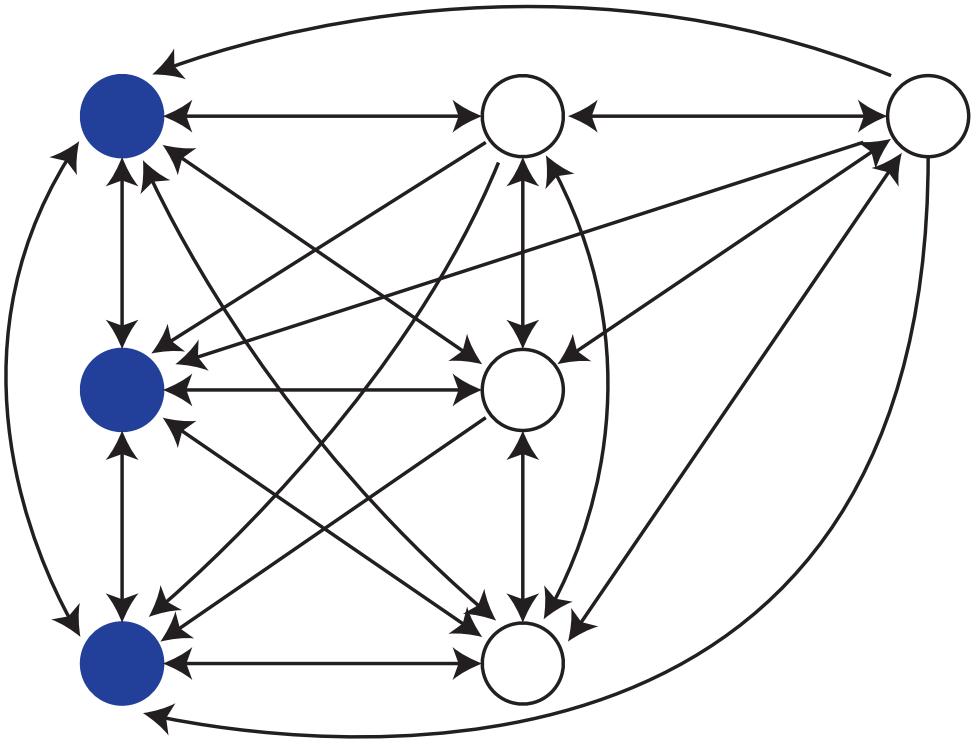
What is the maximum number of arcs in a digraph with n vertices and a given zero forcing number k?

Maximum Arc Digraph



$|E| = 36$, $|\Gamma| = 7$ and $Z(\Gamma) = 3$

Maximum Arc Digraph



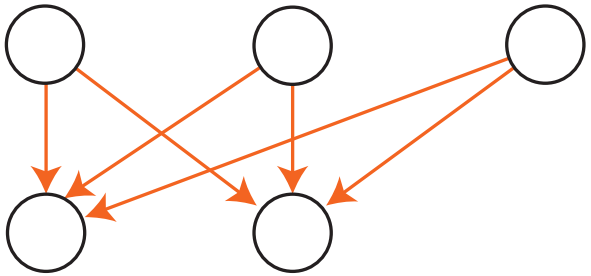
$|E| = 36$, $|\Gamma| = 7$ and $Z(\Gamma) = 3$

$$\sum_{i<j}^{k} n_i n_j + \left(\sum_{i=1}^{k} n_i\right)(k-1) - \binom{k}{2} + \sum_{i=1}^{k}\left[\binom{n_i}{2} + (n_i - 1)\right]$$

Formulation

- Given n_i vertices in the i-th forcing chain and $Z(\Gamma) = k$:

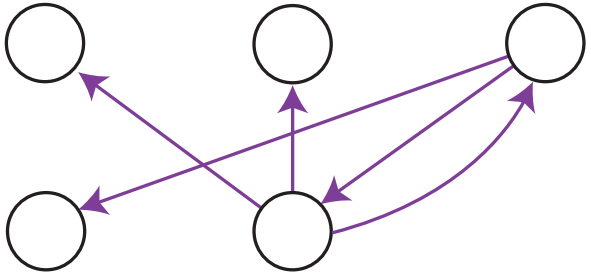
$$\sum_{i<j}^{k} n_i n_j$$



Formulation

- Given n_i vertices in the i-th forcing chain and $Z(\Gamma) = k$:

$$+\left(\sum_{i=1}^{k} n_i\right)(k-1)$$



Formulation

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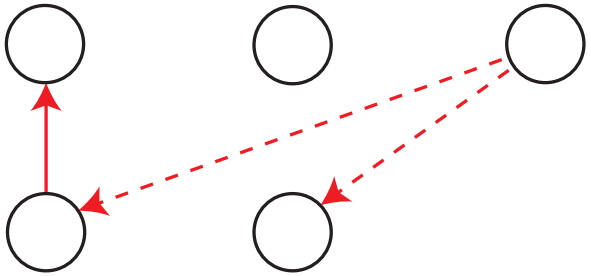
$$-\binom{k}{2}$$



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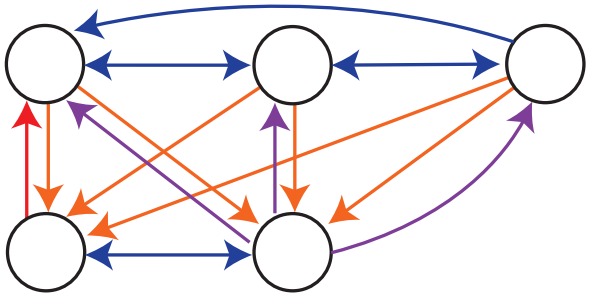
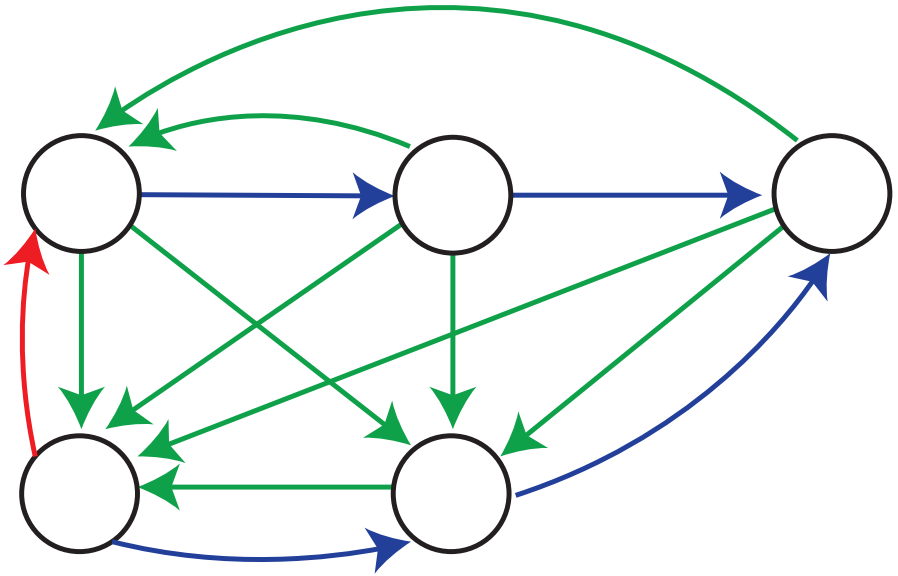


Theorem

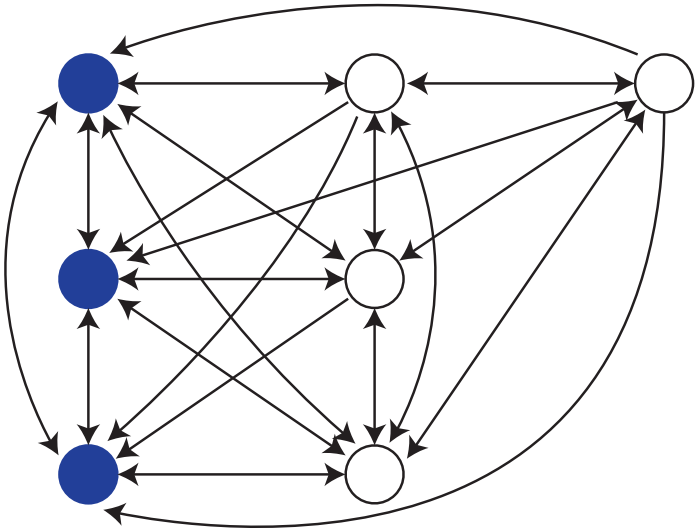
For a digraph Γ of order n with $Z(\Gamma) = k$,

$$|E| \leqslant \binom{n}{2} - \binom{k}{2} + k(n-1)$$

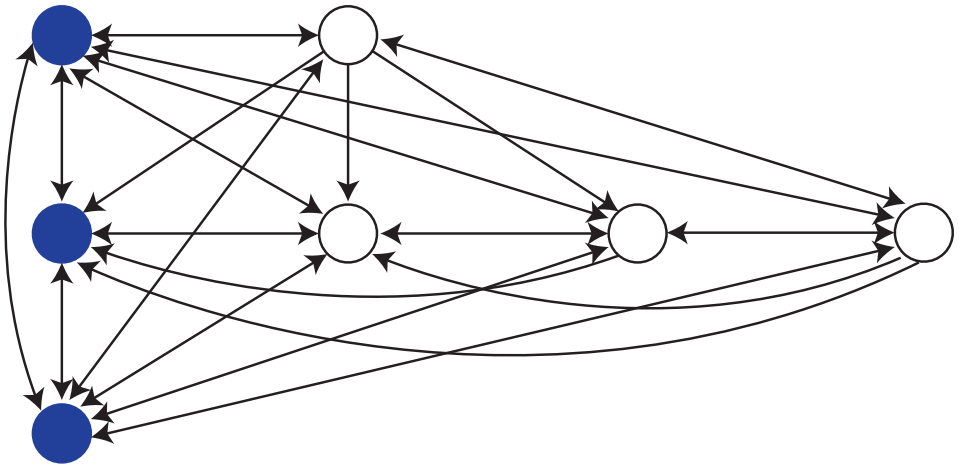


Independence of Distribution of Vertices

Given $|\Gamma| = n$ and $Z(\Gamma) = k$, the maximum number of arcs is independent of the distribution of the vertices into each of the k forcing chains.



$|\Gamma| = 7$ and $Z(\Gamma) = 3$
$|E(\Gamma)| = 36$

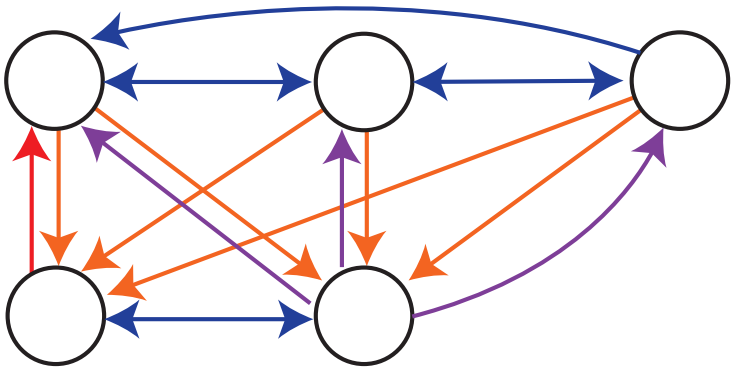


$|\Gamma| = 7$ and $Z(\Gamma) = 3$
$|E(\Gamma)| = 36$

Maximum Nullity of a Maximum Arc Digraph

Theorem

If Γ is a digraph with the maximum number of arcs (by our construction), then $M(\Gamma) = Z(\Gamma)$.



Γ realizing the maximum number of arcs

$$\begin{pmatrix} ? & * & * & 0 & 0 \\ * & ? & * & * & * \\ * & * & ? & * & 0 \\ * & * & * & ? & * \\ * & * & * & * & ? \end{pmatrix}$$

Family of matrices corresponding to Γ

Here $Z(\Gamma) = 2$ and $M(\Gamma) = 2$.

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Acknowledgments

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