

A New Way to Think About Triangles

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1 Background and Motivation

Around 500-600 B.C., either Thales of Miletus or Pythagoras of Samos introduced the Western world to the forerunner of Western geometry. After a good deal of work had been devoted to the field, Euclid compiled and wrote *The Elements* circa 300 B.C. In this work, he gathered a fairly complete backbone of what we know today as Euclidean geometry. For most of the 2,500 years since, mathematicians have used the most basic tools to do geometry—a compass and straightedge. Point by point, line by line, drawing by drawing, mathematicians have hunted for visual and intuitive evidence in hopes of discovering new theorems; they did it all by hand. Judging from the complexity and depth from the geometric results we see today, it's safe to say that geometers were certainly not suffering from a lack of technology.

In today's technologically advanced world, a strenuous effort is not required to transfer the capabilities of a standard compass and straightedge to user-friendly software. One powerful example of this is Geometer's Sketchpad. This program has allowed mathematicians to take an experimental approach to doing geometry. With the ability to create constructions quickly and cleanly, one is able to see a result first and work towards developing a proof for it afterwards. Although one cannot claim proof by empirical evidence, the task of finding interesting things to prove became a lot easier with the help of this insightful and flexible visual tool.

One of the most incredible features of Geometer's Sketchpad is its ability to produce dynamic constructions. In other words, one can create a construction with interdependent elements, alter one of the elements, and observe the resulting movement in the rest of the construction. These motions could not be easily represented otherwise (without the use of a flip book and a huge number of painstakingly accurate drawings). Because of this, Geometer's Sketchpad has enabled modern mathematicians to ask questions that their predecessors were not able to formulate. On paper, geometry is static. With Geometer's Sketchpad, it becomes dynamic.

With this tool in hand, let us begin our project with a few questions: Is there a way to naturally relate triangles? Is there a logical method to define "families" using intrinsic properties of triangles? Can we easily construct these families? What would the triangle space that results from this construction look like topologically? How could we characterize motions through this space? In short, our task is to find a natural and meaningful way to name and relate triangles.

Recognizing there are many ways to approach this task, we need a starting point. Since we want to find a way to characterize families of triangles that utilizes intrinsic triangle properties, it seems logical to begin with triangle centers. Thus, let us start with a brief overview.

2 Basic Triangle Concepts

First, we should define a few very well-known triangle centers. Although we are not including any proofs of their existence in our paper, these are simple to reproduce and can be found in any basic geometry text.

Definition 1. A median is straight line joining a triangle vertex to the midpoint of the opposite side.

Proposition 1. *The three medians intersect at a point. This point is known as the centroid and will be denoted by G .*

Definition 2. A perpendicular bisector is a straight line that bisects a line segment at a right angle.

Definition 3. The circumcircle is the circle determined by a triangle's three vertices.

Definition 4. The circumradius is the radius of the circumcircle.

Definition 5. The circumcenter is the center of the circumcircle, denoted by O henceforth.

Proposition 2. *The three perpendicular bisectors of a triangle meet at the circumcenter.*

Definition 6. An altitude is a straight line going through a triangle vertex, perpendicular to the opposite side or extension of the opposite side.

Proposition 3. *The three altitudes of a triangle intersect at a point. This point is known as the orthocenter and will be denoted by H .*

In addition to the few presented here, there are a huge number of known triangle centers, many of which are constructed through quite elaborate processes. Interestingly, many of these triangle centers are collinear, lying on a line known as the Euler line. A few of these include less common centers such as the nine-point center, the de Longchamps point, the Schiffler point, the Exeter point, and the far-out point. Most importantly for our purposes, the centroid, circumcenter, and orthocenter all lie on the Euler line. Additionally, it is a well-known fact that G always lies $\frac{1}{3}$ of the way from O to H , a relationship of which we will frequently make use. Because of its connection to triangle centers (and the fact that April 15th 2007 is Euler's 300th birthday), it seems natural to use the Euler line as the basis of our construction. Such a construction would take full advantage of much work already done, since it exploits the work already done on triangle centers.

3 Classifying Triangles Based on the Euler line

Now, we can begin our search for a method of constructing a triangle based on a given Euler line. In the end, we want to develop a general method to construct all possible triangles. So, we begin by classifying all triangles that share the same Euler line. Consider the following question: Given the circumcenter, the centroid, and one vertex of a triangle, can you construct a triangle? Is the triangle with these three points unique? Are there other triangles with different vertices but the same circumcenter and centroid? In order to address the first question, we need the known result below. Our own proof of it is provided.

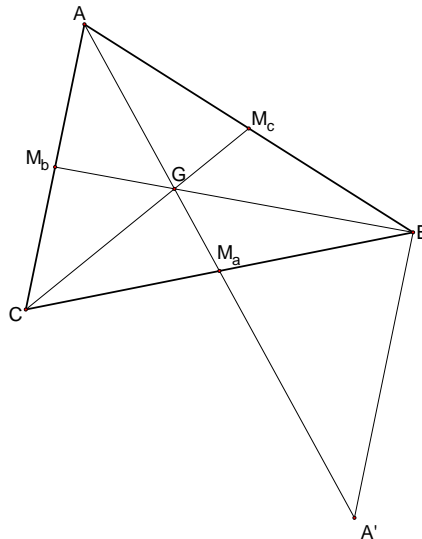


Figure 1: An illustration of Proposition 4.

Proposition 4. *Given $\triangle ABC$ and its centroid G , let M_a be the midpoint of segment \overline{BC} , M_b the midpoint of segment \overline{AC} , and M_c the midpoint of segment \overline{AB} . Then, $AG = 2GM_a$, $BG = 2GM_b$, and $CG = 2GM_c$.*

We will prove this statement for segment \overline{BG} . The proofs for segments \overline{AG} and \overline{CG} are identical.

Proof. Extend segment $\overline{AM_a}$ past M_a and mark point A' such that $A'M_a = AM_a$. Since by construction $\overline{BM_a} \cong \overline{M_aC}$ and by the Vertical Angle Theorem $\angle BM_aA' \cong \angle CM_aA$, it follows from SAS Congruence that $\triangle A'M_aB \cong \triangle AM_aC$. Thus, $\angle BA'M_a \cong \angle CAM_a$. By Proposition 27 of Euclid's *Elements*, this implies that $\overline{AC} \parallel \overline{BA'}$. So, $\angle AM_bG \cong \angle A'BG$. Again by the Vertical Angle Theorem, $\angle AGM_b \cong \angle A'GB$. Thus, $\triangle BA'G \sim \triangle M_bAG$ by AAA Similarity. Since $BA' = AC = 2AM_b$, this and the previous triangle similarity imply that $\frac{BG}{GM_b} = \frac{A'B}{AM_b} = \frac{AC}{AM_b} = 2$. Therefore, $BG = 2GM_b$. \square

With this information in hand, we will show that with a given circumcenter, centroid, and one vertex of a triangle, we can construct a triangle.

3.1 Triangle Construction

Let line l and a point O on l be given. Choose a point A not on l and create circle c centered at O with radius OA . O will be the circumcenter of our triangle and A one of its vertices. Thus, the other two vertices of our triangle will also lie on circle c . Let M and N be the intersections of l with c . Now, choose a point $G \neq O$ on l such that G lies between M and N . G will be the centroid of our triangle. Construct point H on l such that $OG = \frac{1}{3}OH$. By construction, H must be the orthocenter of the triangle. Now, construct the line that passes through A and G and the line that passes through A and H . Since it was just proven that the distance from one vertex of a triangle to its centroid is twice the distance from the centroid to the midpoint of the opposite side, construct point M_a on \overline{AG} such that $GM_a = \frac{1}{2}GA$ and such that M_a is not between A and G . Then, construct line m through M_a such that $m \perp \overline{AH}$. Let B and C be the intersections of m with circle c . Since H is the orthocenter of the triangle, \overline{AH} must be the altitude from vertex A . Moreover, we constructed line m to be perpendicular to \overline{AH} . Thus, line m must be the side of the triangle opposite vertex A , and B and C will be the triangle's remaining two vertices. Therefore, we have constructed a $\triangle ABC$ from a given Euler line l , two points on l , and a point A not on l .

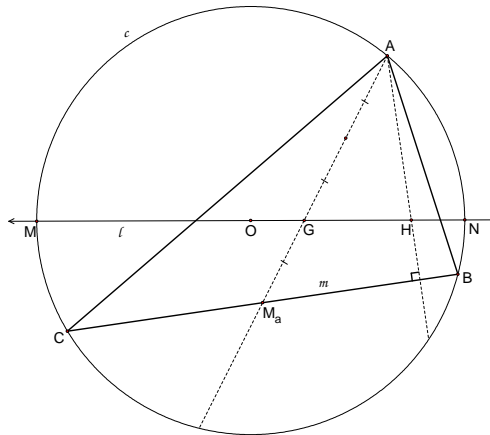


Figure 2: The elements of the triangle construction.

Removing the condition that the specific point A must be a vertex of the triangle produces an infinite number of triangles that have circumcenter O and orthocenter H . So, we will choose to classify triangles according to their circumcenter, centroid, and a given circumradius. As a convention, we will orient ourselves with the Euler line as the traditional x-axis unless otherwise stated.

Definition 7. Let $\triangle ABC$ with circumcenter O and centroid G be given, and let \mathbb{T} be the space of all triangles. Then, define a mapping $F : \mathbb{T} \rightarrow \mathbb{R}^3$, where $F(\triangle ABC) = (\theta, g, r)$ with θ , g , and r determined as follows:

$$\begin{aligned} \theta &= m\angle GOP, \text{ where vertex } P \text{ lies on the "one-vertex side",} \\ g &= OG, \\ r &= OA. \end{aligned}$$

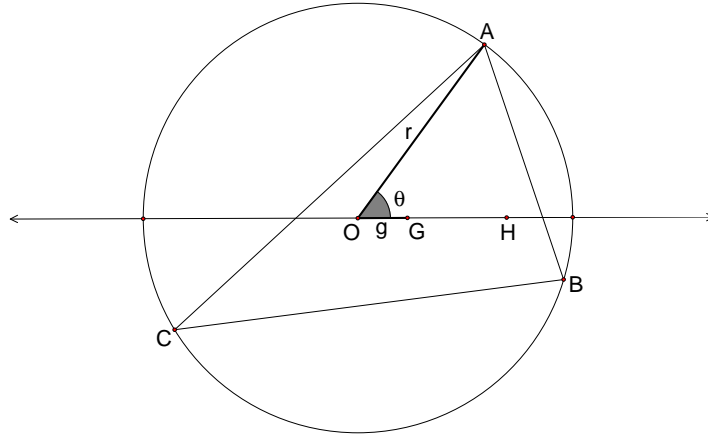


Figure 3: $\overline{OG} = g$, $\overline{OA} = r$, and $\angle GOP = \theta$ where P is the vertex on the “one-vertex” side; in the diagram above, $P = A$

Note that in the event that one vertex falls on the Euler line we will have two “one-vertex sides,” and in that case θ will be the smaller of the angles.

3.2 Special Cases of the Construction

The natural question to ask now is, “is this construction well-defined?” Before we address this, we must first discuss some special cases of our construction. These will help us prove that the construction is in fact well-defined.

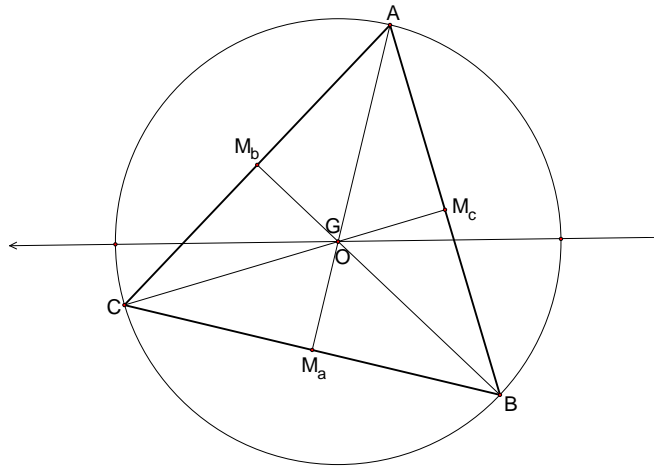


Figure 4: When $g = 0$, O and G are coincident.

Proposition 5. $\triangle ABC$ is equilateral if and only if $g = 0$.

Proof. (\Rightarrow) Let $\triangle ABC$ be equilateral. Let M_a and M_b be the midpoints of sides \overline{BC} and \overline{AC} , respectively. Then, construct lines $\overline{AM_a}$ and $\overline{BM_b}$. Since $\angle ACB \cong \angle ABC$, $\overline{BM_a} \cong \overline{CM_a}$, and $\overline{AB} \cong \overline{AC}$, it follows from SAS congruence that $\triangle ABM_a \cong \triangle ACM_a$. Therefore, $m\angle AM_aB = m\angle AM_aC = \pi$. Thus, $\overline{AM_a}$ is the perpendicular bisector of \overline{BC} . Consequently, the circumcenter O of $\triangle ABC$ lies on $\overline{AM_a}$. Moreover, by construction, the centroid G of $\triangle ABC$ lies on $\overline{AM_a}$.

By the same argument, O and G also lie on $\overline{BM_b}$. Since $\overline{AM_a}$ and $\overline{BM_b}$ only intersect once, the only way that this can occur is if $O = G$. Thus, $OG = g = 0$.

(\Leftarrow) Let $\triangle ABC$ be given with $OG = 0$. Let M_a and M_b be defined as above. Now, construct \overline{AG} . Since G is the centroid, M_a must lie on \overline{AG} . Since $O = G$, $\angle AM_aB \cong \angle AM_aC$. Thus, since $\overline{AM_a}$ is common, by SAS congruence, $\triangle AM_aB \cong \triangle AM_aC$. This implies that $\overline{AB} \cong \overline{AC}$. By the same argument, $\overline{BC} \cong \overline{AB}$. Therefore, $\overline{AB} \cong \overline{AC} \cong \overline{BC}$, and $\triangle ABC$ is equilateral. \square

Note that an equilateral triangle does not in fact have an Euler line, since the triangle centers determining the Euler line collapse a single point. Also, since G and O are the same point the angle formed by O , G , and a vertex does not technically exist. Thus, we will define $\theta = \frac{\pi}{2}$ for reasons that will become clear later.

Proposition 6. *Let $\triangle ABC$ with orthocenter H be given. If H lies on the circumcircle of $\triangle ABC$, then H is coincident with one vertex of $\triangle ABC$.*

Proof. (\Rightarrow) Let circle c centered at O be given, and let H and A be points on c such that $H \neq A$. We will construct $\triangle ABC$ with circumcenter O , circumcircle c , and orthocenter H and show that H must coincide with either B or C .

To begin, connect A to the centroid G , which lies on the Euler Line one third of the way from O to H . Extend \overline{AG} beyond G one-half the length of \overline{AG} , and call the endpoint of this extended segment M_a . By Proposition 4, M_a must be the midpoint of \overline{BC} . Since H is the orthocenter of $\triangle ABC$, \overline{AH} must be perpendicular to \overline{BC} . Thus, if we construct the line $\overline{M_aP}$, where P is the point on \overline{AH} such that $\overline{M_aP} \perp \overline{AH}$, this line will contain vertices B and C . We will show that $P = H$.

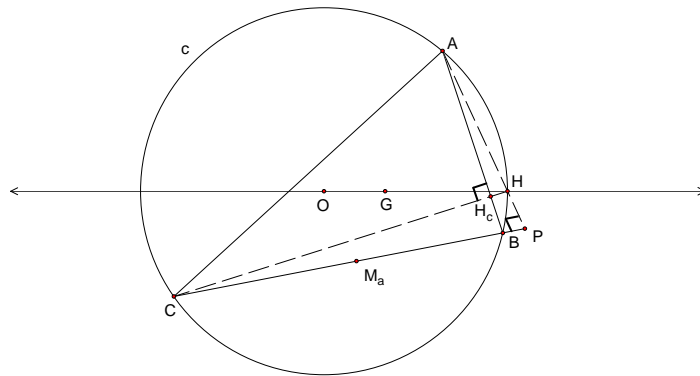


Figure 5: P lies past H on ray \overrightarrow{AH} .

To do this, suppose to the contrary that $P \neq H$. Then, P lies between A and H , past H on the ray \overrightarrow{AH} , or past A on the ray \overrightarrow{HA} .

Case 1: (Refer to Figure 5 above.) Suppose P lies past H on \overrightarrow{AH} . Then, vertex B , the intersection of the circumcircle and the line $\overline{M_aP}$ lies on the opposite side of the Euler line from

A. By construction, $m\angle APC = \frac{\pi}{2}$. Now, construct \overline{CH} . \overline{CH} is the altitude from vertex C and thus meets \overline{AB} perpendicularly at H_c . Consider $\triangle ACH_c$. By angle subtraction, $m\angle ACH + m\angle BAC = \frac{\pi}{2}$. Consider $\triangle ACP$. Again by angle subtraction, $m\angle ACP + m\angle CAP = \frac{\pi}{2}$. However, $\frac{\pi}{2} = m\angle ACP + m\angle CAP > m\angle ACH + m\angle BAC = \frac{\pi}{2}$. This is a contradiction. Thus, P cannot lie past H on \overline{AH} .

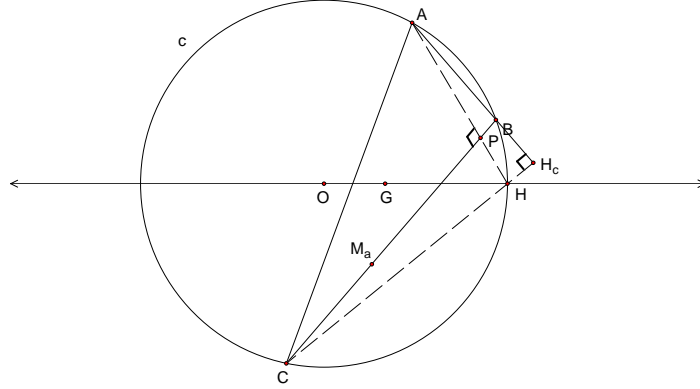


Figure 6: P lies between A and H .

Case 2: (Refer to Figure 6 above.) Suppose P lies between A and H . Then, vertex B , the intersection of the circumcircle and the line $\overline{M_aP}$ lies on the same side of the Euler line as A . As in Case 1, since \overline{AH} is the altitude from A , $m\angle APC = \frac{\pi}{2}$. Thus, $m\angle CAP + m\angle ACP = \frac{\pi}{2}$. Construct \overline{CH} . Since \overline{CH} is the altitude from C to \overline{AB} , it meets \overline{AB} perpendicularly at H_c . It follows that $m\angle CAH_c + m\angle ACH_c = \frac{\pi}{2}$. However, $\frac{\pi}{2} = m\angle CAP + m\angle ACP < m\angle CAH_c + m\angle ACH_c = \frac{\pi}{2}$. This is a contradiction. Thus, P cannot lie between A and H .

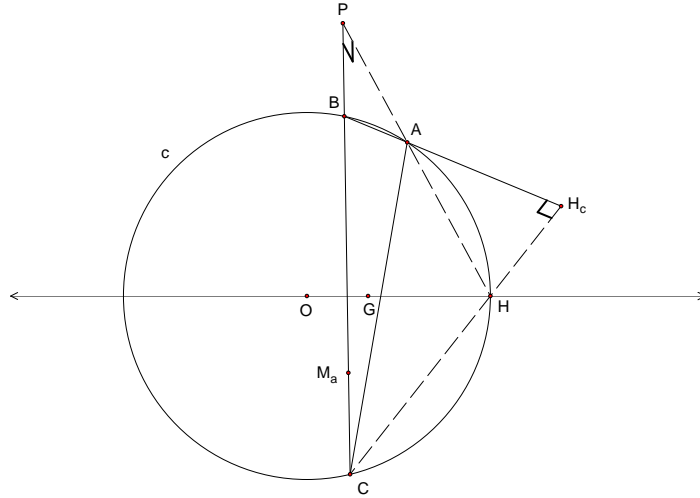


Figure 7: P lies past A on ray \overline{HA} .

Case 3: (Refer to Figure 7 above.) Suppose P lies past A on the ray \overline{HA} . As in the previous two cases, $m\angle APC = \frac{\pi}{2}$. Thus, $m\angle BAP + m\angle ABP = \frac{\pi}{2}$. This implies that $m\angle ABC = m\angle BAP + \frac{\pi}{2} > \frac{\pi}{2}$. Construct \overline{CH} . Since \overline{CH} is the altitude from C to \overline{AB} , it meets \overline{AB} perpendicularly at H_c . Thus, $m\angle H_cBC = m\angle ABC < \frac{\pi}{2}$. This is a contradiction. Thus, P cannot lie past A on ray \overline{HA} .

Therefore, P is coincident with H . By the initial discussion, $\overline{M_aH}$ contains the vertices B and C of $\triangle ABC$. Since these vertices must lie on the circumcircle c , and H is on the circumcircle, it

follows that H must coincide with either B or C . \square

Corollary 1. *Let $\triangle ABC$ with orthocenter H be given. Then, H is coincident with vertex B if and only if $m\angle ABC = \frac{\pi}{2}$.*

Proof. (\Rightarrow) Assume that H is coincident with vertex B . It follows immediately from the work in Proposition 6 that $m\angle ABC = \frac{\pi}{2}$.

(\Leftarrow) Assume that $m\angle ABC = \frac{\pi}{2}$. Then, \overline{BC} is the altitude from vertex C to \overline{AB} , and \overline{AB} is the altitude from vertex A to \overline{BC} . Since the altitudes of a triangle coincide at H , it follows that B and H are coincident. \square

Proposition 7. *Triangle $\triangle ABC$ is a right triangle if and only if $g = \frac{r}{3}$.*

Proof. (\Leftarrow) If $g = \frac{r}{3}$, then $OH = r$. That is, H lies on the circumcircle. Therefore, by Proposition 6 and Corollary 1, $\triangle ABC$ is a right triangle.

(\Rightarrow) If $\triangle ABC$ is a right triangle, then by Corollary 1, H is coincident with one vertex. Without loss of generality, let that vertex be vertex B . Thus, H is on the circumcircle, so $OH = r$. Since G is one-third of the way from O to H , it follows that $g = \frac{r}{3}$. \square

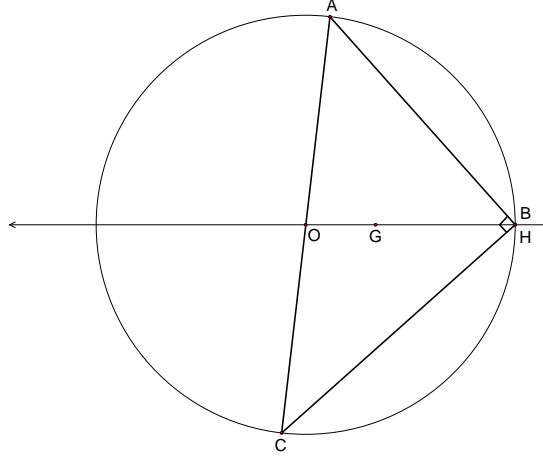


Figure 8: Right triangles occur when $g = \frac{r}{3}$ and H lies on the circumcircle.

To follow Proposition 7, we have the next two propositions which deal with the cases in which $g > \frac{r}{3}$ and $g < \frac{r}{3}$.

Proposition 8. *Triangle $\triangle ABC$ is an obtuse triangle if and only if $g > \frac{r}{3}$.*

Proof. We will begin with the reverse direction.

(\Leftarrow) Let $g > \frac{r}{3}$. Then, since $OH = 3OG = 3g > 3(\frac{r}{3}) = r$, it follows that the orthocenter H lies outside the circumcircle. Construct the altitude from vertex A . Without loss of generality, assume that H lies past vertex B on ray \overrightarrow{CB} . Since $m\angle AHB = \frac{\pi}{2}$, it follows that $m\angle ABH < \frac{\pi}{2}$. Thus, $m\angle ABC > \frac{\pi}{2}$. Therefore, $\triangle ABC$ is obtuse.

(\Rightarrow) Let $\triangle ABC$ be obtuse. Without loss of generality, assume that $m\angle ABC > \frac{\pi}{2}$. Construct the altitude from vertex A . It will intersect side \overline{BC} either past B on ray \overrightarrow{CB} or past C on ray \overrightarrow{BC} by the following: Suppose that the altitude from vertex A intersects segment \overline{BC} . Let H_a be its intersection with \overline{BC} . Then, $m\angle AH_aB = \frac{\pi}{2}$. By assumption, $m\angle ABC > \frac{\pi}{2}$. Thus, $m\angle AH_aB + m\angle ABC + m\angle BAH_a > \pi$. This is a contradiction. Thus, the altitude from vertex A

intersects an extension of segment \overline{BC} . By the same argument, the altitude from vertex C intersects an extension of segment \overline{AB} . Now, the extensions of \overline{BC} and \overline{AB} must lie outside the circumcircle of $\triangle ABC$. Therefore, the orthocenter H , the intersection of the altitudes from vertices A and C , must lie outside the circumcircle. Since $OH = 3OG = 3g$, and since we now know that $OH > r$, it follows that $3g > r$ or $g > \frac{r}{3}$. \square

Proposition 9. *Triangle $\triangle ABC$ is an acute triangle if and only if $g < \frac{r}{3}$.*

Proof. This follows from a process of elimination using Propositions 7 and 8. \square

The three propositions above are nice because they indicate that the construction cleanly separates acute, right, and obtuse triangles. We will next examine another type of triangles — isosceles triangles.

Proposition 10. *If $\triangle ABC$ is isosceles, one of its vertices lies on the Euler line.*

Proof. Assume that $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{AC}$. Let M_a be the midpoint of \overline{BC} . Then, $\overline{AM_a}$ contains the centroid G of $\triangle ABC$. Moreover, since $\overline{AB} \cong \overline{AC}$, $\overline{BM_a} \cong \overline{M_aC}$, and $\overline{AM_a}$ is shared, it follows that $\triangle AM_aB \cong \triangle AM_aC$. Thus, $\angle AM_aB \cong \angle AM_aC$. Since they are supplementary, $m\angle AM_aB = m\angle AM_aC = \frac{\pi}{2}$. This implies that $\overline{AM_a}$ is the altitude of $\triangle ABC$ from point A . Thus, the orthocenter H of $\triangle ABC$ must lie on $\overline{AM_a}$. Therefore, $\overline{AM_a}$ coincides with the Euler line. \square

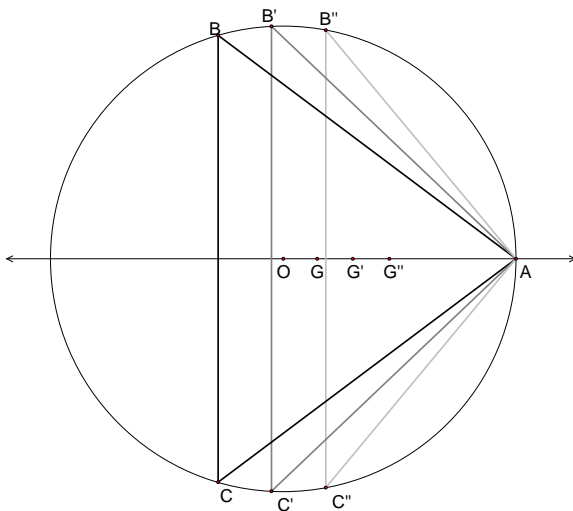


Figure 9: A few different isosceles triangles.

Proposition 11. *If $\triangle ABC$ is not a right triangle and has a vertex lying on the Euler line, then it is isosceles.*

Proof. Assume that the Euler line l of $\triangle ABC$ intersects vertex A . Since the centroid of $\triangle ABC$ lies on l , it follows that l must intersect \overline{BC} at the midpoint M of \overline{BC} . Thus, $\overline{BM_a} \cong \overline{CM_a}$. Also, the orthocenter of $\triangle ABC$ lies on l and is not coincident with A . Thus, $m\angle AM_aB = m\angle AM_aC = \frac{\pi}{2}$. Finally, by SAS congruence, $\triangle AM_aB \cong \triangle AM_aC$ (since side $\overline{AM_a}$ is shared). This implies that $\overline{AB} \cong \overline{AC}$. \square

3.3 Showing that the Construction is Well-Defined

We are finally ready to show that the construction is well-defined.

Proposition 12. *The mapping $F : \mathbb{T} \rightarrow \mathbb{R}^3$ is well-defined.*

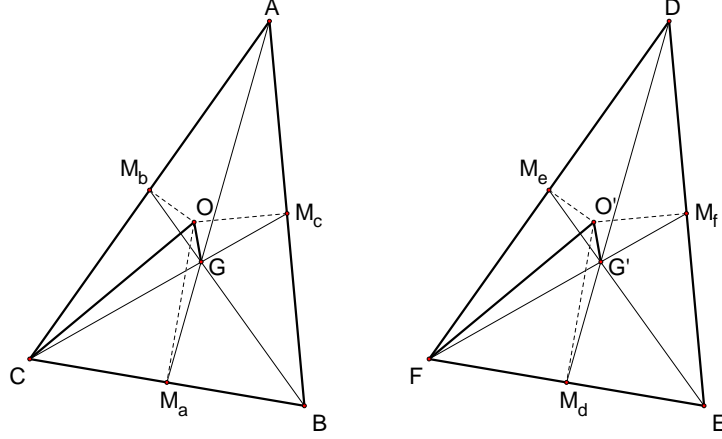


Figure 10: Showing the construction is well-defined.

Proof. To show our construction is well-defined we must show that given two triangles $\triangle ABC$ and $\triangle DEF$, $\triangle ABC \cong \triangle DEF$ if and only if the coordinates (θ, g, r) of $\triangle ABC$ equal the coordinates (θ', g', r') of $\triangle DEF$. The reverse direction is covered since given two sets of equal parameters, we know that the corresponding triangles will be congruent by construction. Thus, we only have to show that two congruent triangles will produce the same parameters.

Assume $\triangle ABC \cong \triangle DEF$ with A corresponding to D , B corresponding to E , and C corresponding to F . Construct the centroids and circumcenters of the given triangles. Since the triangles are congruent it must be that corresponding vertices lie on the same side of the Euler line with respect to the other vertices.

Case i) Without loss of generality let A and B be on the same side of the Euler line. Then, D and E must also lie on the same side of the Euler line. Also, it must be that $r = r'$, since otherwise the triangles would have different circumcircles.

Therefore, $\overline{OC} \cong \overline{O'F}$ in the diagram. Also we know the corresponding medians must be congruent by simple SAS arguments. Now, $\overline{GC} = \frac{2\overline{CM}_c}{3}$ and $\overline{G'F} = \frac{2\overline{FM}_f}{3}$. But $\overline{CM}_c \cong \overline{FM}_f$ because they are corresponding medians. Therefore, $\overline{GC} \cong \overline{G'F}$. By a similar argument, $\overline{GM}_a \cong \overline{G'M_d}$. So, by SSS $\triangle CGM_a \cong \triangle FG'M_d$, which gives us $\angle GCM_a \cong \angle G'FM_d$. Also, by a hypotenuse-leg right-triangle congruence argument, $\triangle OCM_a \cong \triangle O'FM_d$. Therefore, $\angle OCM_a \cong \angle O'FM_d$. By subtraction, $\angle GCO \cong \angle G'FO'$. So, by ASA congruence, $\triangle GCO \cong \triangle G'FO'$ and $\overline{GO} \cong \overline{G'O'}$. Thus, it must be that $g = g'$. It also follows that $\angle GOC \cong \angle G'O'F$, so $\theta \cong \theta'$.

We now have to consider the cases in which we have a vertex on the Euler line. First, we will consider the right triangle case.

Case ii) If we let $m\angle ABC = \frac{\pi}{2} = m\angle DEF$, then $r = \frac{AC}{2} = \frac{DF}{2} = r'$. Since we have right triangles, $g = \frac{r}{3} = g'$. Also $\triangle OAB \cong \triangle O'DE$ by SSS. So $\angle BOA \cong \angle EO'D$. If $\angle BOA \leq \frac{\pi}{2}$, then $\theta = m\angle BOA = \theta'$. Otherwise, $\angle BOA > \frac{\pi}{2}$, which means $\theta = \pi - m\angle BOA = \theta'$.

Case iii) If we have congruent isosceles triangles, the arguments for cases i) and ii) hold, except that it does not matter which vertex you choose to form θ as long as it is not the vertex on the Euler line.

Case iv) When we have congruent equilateral triangles, r must equal r' , θ (likewise θ') does not actually exist or have any affect in determining the triangle and $g = 0 = g'$. \square

4 The Topology of Triangle Space

As mentioned in the introduction, one of our goals is to figure out what triangle space looks like. One could argue that this can be done by considering triangles in the traditional sense: defining a triangle by the lengths of its sides. In either case, you are given three numbers, and those three numbers correspond to a point in \mathbb{R}^3 . This point corresponds to the triangle defined by the three numbers. The problem with considering triangles in the traditional sense is that every triangle is corresponds to six different points (one for each permutation of the three sides). It is not a trivial problem to define the space in such a way that every triangle is represented only once. By defining a triangle the way we have, we have ensured that every triangle is represented by exactly one ordered triplet. However, not every point in \mathbb{R}^3 represents a triangle. It will be our next task to determine exactly which values of θ , g , and r give us legitimate triangles.

To help us reach our goal, let us first examine similar triangles.

Theorem 1. *Two triangles $\triangle ABC$ defined by (θ, g, r) and $\triangle A'B'C'$ defined by (θ', g', r') are similar if and only if $\theta = \theta'$, $g = kg'$, and $r = kr'$ for $k > 0$.*

Proof. (\Leftarrow) Let $\triangle ABC$ be similar to $\triangle A'B'C'$.

Let G and G' be the centroids of and O and O' the circumcenters of $\triangle ABC$ and $\triangle A'B'C'$, respectively. Also, let M_a and M_a' be the midpoints of segments \overline{BC} and $\overline{B'C'}$, respectively. Let M_b and M_b' be the midpoints of \overline{AC} and $\overline{A'C'}$, respectively. Since the two triangles are similar, it follows that $\frac{AB}{A'B'} = \frac{BM_a}{B'M_a'} = k$. Therefore, $\triangle ABM_a \sim \triangle A'B'M_a'$. So, $\frac{AM_a}{A'M_a'} = k$. Now, G and G' must lie on $\overline{AM_a}$ and $\overline{A'M_a'}$, respectively. Moreover, $GM_a = \frac{1}{3}AM_a$ and $G'M_a' = \frac{1}{3}A'M_a'$. So, $\frac{GM_a}{G'M_a'} = \frac{AM_a}{A'M_a'}$. Also, $\frac{BC}{B'C'} = \frac{M_bC}{M_b'C'} = k$. So, $\triangle BCM_b \sim \triangle B'C'M_b'$. So, $\frac{BM_b}{B'M_b'} = k$. But, $GM_b = \frac{1}{3}BM_b$ and $G'M_b' = \frac{1}{3}B'M_b'$. Thus, $\frac{GM_b}{G'M_b'} = k$. This implies that $\triangle BM_bM_a \sim \triangle G'M_b'M_a'$ by SSS similarity. Thus, $\angle GM_bM_a \cong \angle G'M_b'M_a'$. Now,

$$\begin{aligned} m\angle O'M_b'G' + m\angle G'M_b'M_a' + m\angle M_a'M_b'C' &= \pi \\ m\angle GM_bM_a &= m\angle G'M_b'M_a' \\ m\angle M_aM_bC &= m\angle BM_bC - m\angle GM_bM_a = m\angle B'M_b'C' - m\angle G'M_b'M_a' = m\angle M_a'M_b'C' \\ m\angle OM_bG &= m\angle O'M_b'G' \end{aligned}$$

Thus, by SAS similarity, $\triangle OM_bG \cong \triangle O'M_b'G'$. This implies that $\frac{OG}{O'G'} = \frac{GM_b}{G'M_b'} = k$. So, $OG = kO'G'$. But, $O'G' = g'$ and $OG = g$. Thus, $g = kg'$.

Now, we know that $\frac{M_bC}{M_b'C'} = \frac{M_aC}{M_a'C'} = k$. So, $\triangle M_bM_aC \sim \triangle M_b'C'M_a'$ by SAS similarity. Therefore, $\angle M_b'M_a'C' \cong \angle M_bM_aC$, $\angle M_aM_bC \cong \angle M_a'M_b'C'$, and $\frac{MM_a}{M'M_a'}$. Thus, by subtraction, $\angle OM_aM \cong \angle O'M_a'M_b'$ and $\angle OM_bM_a \cong \angle O'M_b'M_a'$. This implies that $\triangle OM_bM_a \sim \triangle O'M_b'M_a'$. So, $\frac{OM_b}{O'M_b'} = \frac{M_bM_a}{M_b'M_a'} = k$. Thus, $\triangle OM_bC \sim \triangle O'M_b'C'$. Therefore, $\frac{OC}{O'C'} = k$. But, $OC = r$ and $O'C' = r'$. So we know $r = kr'$.

Now, construct lines \overline{OG} , $\overline{O'G'}$, and segments \overline{OC} and $\overline{O'C'}$. We have shown that the segments connecting the centroids to the vertices are proportional by k , and $OG = kO'G'$. Thus, $\triangle OGC \sim \triangle O'G'C'$ by SSS similarity. This implies that $\angle GOC \cong \angle G'O'C'$. So, since $\angle GOC = \theta$ and $\angle G'O'C' = \theta'$, $\theta = \theta'$.

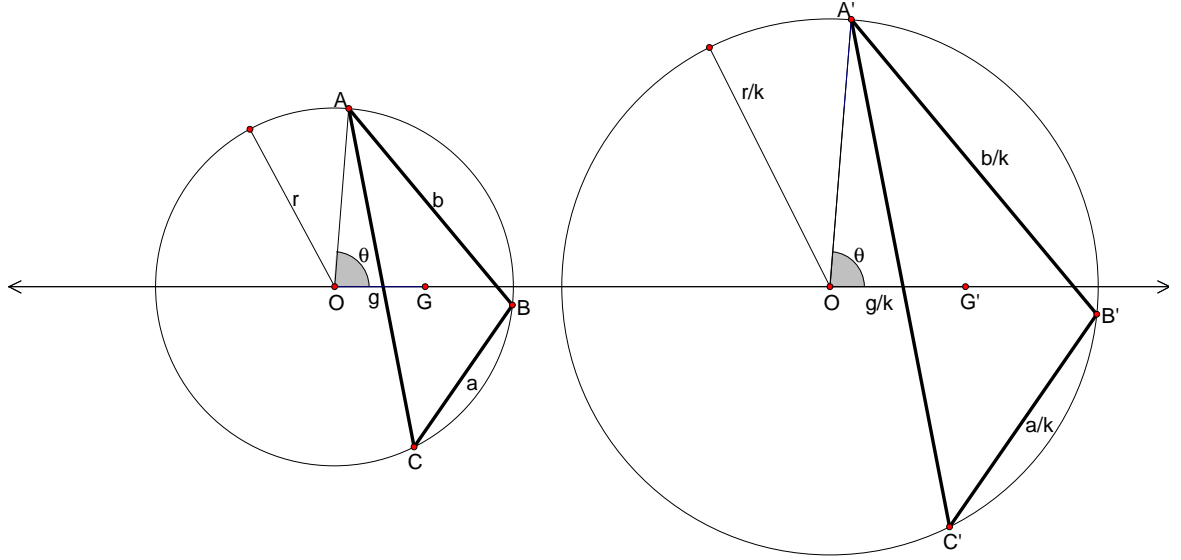


Figure 11: Angle θ is the same in both triangles.

(\Rightarrow) Let $\triangle ABC$ defined by (θ, g, r) and $\triangle A'B'C'$ defined by (θ', g', r') be two triangles with the property that $r = kr'$, $g = kg'$, and $\theta = \theta'$.

First, let M_a be the midpoint of \overline{CB} . Now, we can see that $\triangle AOG \sim \triangle A'O'G'$ by SAS similarity. Thus, $\frac{AG}{A'G'} = k$. So,

$$\frac{AM_a}{A'M'_a} = \frac{\frac{3}{2}AG}{\frac{3}{2}A'G'} = \frac{AG}{A'G'} = k$$

Then, $\frac{GM_a}{G'M'_a} = k$ and $GM_a = kG'M'_a$. Also, $\angle AGO \cong \angle A'G'O'$ and thus $\angle OGM_a \cong \angle O'G'M'_a$. Therefore, $\triangle OGM_a \sim \triangle O'G'M'_a$ by SAS similarity. So, $\frac{OM_a}{O'M'_a} = k$ and $OM_a = kO'M'_a$. Also, $\angle OM_aG \cong \angle O'M'_aG'$. Because O and O' lie on the perpendicular bisectors to sides \overline{BC} and $\overline{B'C'}$ respectively, we know,

$$\begin{aligned} CM_a^2 &= (kr)^2 - OM_a^2 \\ (C'M'_a)^2 &= r'^2 - (O'M'_a)^2 \\ CM_a^2 &= (kr)^2 - (kO'M'_a)^2 \\ CM_a^2 &= k^2(r^2 - (O'M'_a)^2) \\ CM_a^2 &= k^2(C'M'_a)^2 \\ CM_a &= kC'M'_a \end{aligned}$$

Also, $BC = 2CM_a = 2kC'M'_a = kB'C'$. Thus, $\triangle ACM_a \sim \triangle A'C'M'_a$ by SAS similarity. Thus, $\angle ACM_a \cong \angle A'C'M'_a$ and $\triangle ABC \sim \triangle A'B'C'$ by SAS similarity. □

Let us now begin looking at the bounds on θ , g , and r .

Proposition 13. *If (θ, g, r) represents a triangle, then $r > 0$.*

Proof. The only acceptable values of r are $r > 0$ because $r = 0$ would mean that the circumcircle is only a point, which does not allow for any triangles. Moreover, $r < 0$ is absurd. Also, if r is bounded

above, then we restrict ourselves to triangles with side lengths less than $2r$, so r can be any positive number. \square

That being said, Theorem 1 implies that r is just a scaling variable. Therefore we can get a clear picture of what triangle space looks like by examining cross-sections of \mathbb{R}^3 parallel to the θg -plane.

Proposition 14. *If (θ, g, r) represents a triangle, then $0 \leq g < r \frac{\cos[\frac{1}{2}(\theta + \sin^{-1}(\frac{1}{2} \sin \theta))]}{\cos[\frac{1}{2}(\theta - \sin^{-1}(\frac{1}{2} \sin \theta))]}$*

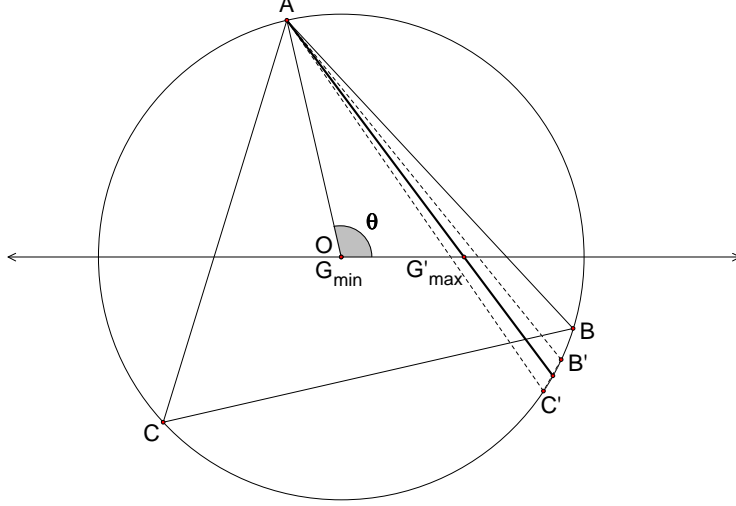


Figure 12: Note that vertices B' and C' are coincident.

Proof. As motivation for this proof, imagine being given a triangle inscribed in a circle. Take the centroid G of the triangle, which lies on the Euler line, and start moving it away from the circumcenter toward the closest edge of the circle. Since G is the intersection of the lines connecting the vertices of the triangle with the midpoints of the opposite sides, it follows that as G approaches the edge of the circle, two vertices of the triangle (say B and C) begin to come closer together. This is because G is approaching one side of the triangle (say \overline{AB}), and as it gets closer and closer to that side, the line connecting vertex A and the midpoint of \overline{BC} , M_a moves closer and closer to the edge of the circumcircle. Thus, when G reaches side \overline{AB} , \overline{BC} and consequently $\triangle ABC$ disappears. It seems that this point is the limit for OG or g . Therefore, we first identify a way to calculate this limit.

Let M_a again be the midpoint of the side opposite vertex A . We want to find the value of g so that M_a lies on the the circumcircle. Now, consider $\triangle OAM_a$. Let A be the vertex of $\triangle ABC$ inscribed in the circle and M_a be the point on ray \overrightarrow{AG} such that $AG = 2GM_a$. Let $y = GM_a$ and $g = OG$.

Then, since $\overline{OM_a}$ and \overline{OA} are both radii of the circle, it follows that $\triangle OAM$ is isosceles. Thus, $\angle OAM_a \cong \angle OM_aA$. Let $\alpha = \angle M_aOG$ and $\angle GOA = \theta$. Now, by the Law of Sines, $\frac{\sin \theta}{2y} = \frac{\sin \angle OAM_a}{g}$ in $\triangle OAG$. Also, $\frac{\sin \alpha}{y} = \frac{\sin \angle OM_aA}{g} = \frac{\sin \angle OAM_a}{g}$ in $\triangle M_aOG$. Therefore, $\frac{\sin \theta}{2y} = \frac{\sin \alpha}{y}$, which implies that $\sin \alpha = \frac{\sin \theta}{2}$. This in turn implies that $\alpha = \sin^{-1} \frac{1}{2} \sin \theta$. Now, $\frac{\pi - (\theta + \alpha)}{2} = \beta$. If we define $\gamma = \angle OGA$, then $\gamma = \pi - \theta - \beta = \pi - \theta - \frac{\pi}{2} + \frac{\theta}{2} + \frac{\alpha}{2} = \frac{\pi}{2} - \frac{\theta}{2} + \frac{\alpha}{2}$. Finally, again by the Law of Sines, $\frac{\sin \frac{\pi}{2} - \frac{\theta}{2} - \frac{\alpha}{2}}{x} = \frac{\sin \frac{\pi}{2} - \frac{\theta}{2} + \frac{\alpha}{2}}{r}$. $\therefore g = r \frac{\sin \frac{\pi}{2} - \frac{\theta}{2} - \frac{\alpha}{2}}{\sin \frac{\pi}{2} - \frac{\theta}{2} + \frac{\alpha}{2}}$. Replacing α with $\sin^{-1} \frac{1}{2} \sin \theta$ and realizing that $\sin(\frac{\pi}{2} - \phi) = \cos \phi$ provides the desired result,

$$g < r \frac{\cos[\frac{1}{2}(\theta + \sin^{-1}(\frac{1}{2} \sin \theta))]}{\cos[\frac{1}{2}(\theta - \sin^{-1}(\frac{1}{2} \sin \theta))]}.$$

Now that we have g bounded above, we consider the lower bound. Clearly we need to include $g = 0$ because we need to include equilateral triangles. We have always looked at our triangles with G on the right side of O . If we look at the triangles we get with G on the left side of O we realize they are the same as the ones with G on the right side. They can be mapped to each other by reflecting across the line perpendicular to the Euler line at O . Thus,

$$0 \leq g < r \frac{\cos[\frac{1}{2}(\theta + \sin^{-1}(\frac{1}{2} \sin \theta))]}{\cos[\frac{1}{2}(\theta - \sin^{-1}(\frac{1}{2} \sin \theta))]} .$$

□

The bounds for θ are a lot harder to sort out because they come in cases which depend on g . For the following propositions, we will denote the θ value for an isosceles triangle with a vertex and G lying on the Euler line on the same side of O as θ_R . Also, we will denote the θ value for an isosceles triangle with a vertex and G lying on the Euler line on the opposite sides of O as θ_L .

Proposition 15. *If $g < \frac{r}{3}$, then $\cos^{-1}(\frac{3g+r}{2r}) \leq \theta \leq \cos^{-1}(\frac{3g-r}{2r})$.*

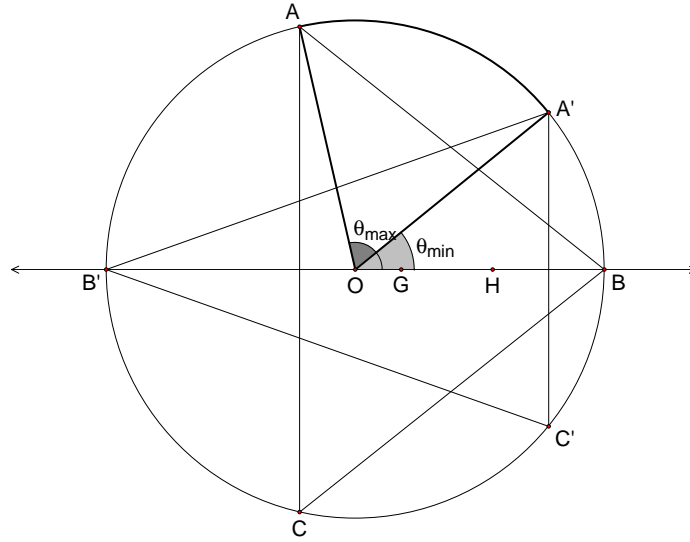


Figure 13: The bounds are created by an isosceles triangle on either end.

Proof. Since $g < \frac{r}{3}$, every triangle with a vertex on the Euler line is an isosceles triangle. Consider the isosceles triangle with vertex B on the Euler line on the same side of O that G is on. So, $\theta_R = m\angle GOA$. Let M_b be the midpoint of \overline{AC} . Since this is an isosceles triangle, M_b lies on the Euler line and $m\angle AM_bO = \frac{\pi}{2}$. Also, $r - g = BG = 2(GM_b)$. Therefore, $OM = \frac{r-3g}{2r}$. We also know $m\angle AOM = \pi - \theta_R$. So $\cos(\pi - \theta_R) = -\cos \theta_R = \frac{r-3g}{2r}$. Therefore,

$$\theta_R = \cos^{-1}\left(\frac{3g-r}{2r}\right) .$$

A consequence of this isosceles triangle case is that $\angle GOA \cong \angle AOB$. Also, $\angle GOA \cong \angle GOC$, and both measure θ , so we can think of $\angle GOA$ as θ . By the law of cosines, we know that

$$\angle AOB = \cos^{-1}\left(\frac{2r^2 - (AB)^2}{2r^2}\right) .$$

In the appendix, we prove that

$$AB = \sqrt{3r^2 - 3rg \cos \theta - 3rg \sin \theta \frac{\sqrt{3r^2 - 9g^2 + 6rg \cos \theta}}{\sqrt{r^2 + 9g^2 - 6rg \cos \theta}}}.$$

When $\theta = \cos^{-1}(\frac{3g-r}{2r})$, (after some simplification) we see

$$\frac{\partial(\theta - m\angle AOB)}{\partial \theta} = 1 - \frac{3g}{r}.$$

So, since $g < \frac{r}{3}$,

$$\frac{\partial(\theta - m\angle AOB)}{\partial \theta} > 0.$$

This means that as we increase $m\angle GOA$, the difference $m\angle GOA - m\angle BOA$ increases. So,

$$m\angle GOA - m\angle BOA > 0.$$

Therefore, A and B must be on the same side of the Euler line. This means that θ is made by $\angle GOC$, and

$$\angle GOC < \cos^{-1}(\frac{3g-r}{2r}),$$

otherwise C and B would be on the same side of the Euler line (and B cannot be on both sides of the Euler line). This proves that θ_R is the upper bound for θ when $g < \frac{r}{3}$.

Likewise, the lower bound for θ is found in the other isosceles triangle case. That is, when we have an isosceles triangle with a vertex (without loss of generality, C) on the opposite side of O from G . Let θ_L be made by $\angle GOA$. Again, let M_c be the midpoint of \overline{AB} . Since this is an isosceles triangle, M_c lies on the Euler line and $m\angle AM_cO = \frac{\pi}{2}$. Also, $g+r = CG = 2(GM_c)$. Therefore, $OM_c = \frac{3g+r}{2r}$. We also know $m\angle AOG = \theta_L$. So,

$$\theta_L = \cos^{-1}(\frac{3g+r}{2r}).$$

The consequence of this isosceles triangle case is that $\angle GOA + \angle AOC = \pi$. Also, $\angle GOA \cong \angle GOB$, and both measure θ , so we can think of $\angle GOA$ as θ . By the law of cosines, we know that

$$\angle AOC = \cos^{-1}(\frac{2r^2 - (AC)^2}{2r^2}).$$

In the appendix, we prove that

$$AC = \sqrt{3r^2 - 3rg \cos \theta + 3rg \sin \theta \frac{\sqrt{3r^2 - 9g^2 + 6rg \cos \theta}}{\sqrt{r^2 + 9g^2 - 6rg \cos \theta}}}.$$

When $\theta = \cos^{-1}(\frac{3g+r}{2r})$, (after some simplification) we see

$$-\frac{\partial(\theta + m\angle AOC - \pi)}{\partial \theta} = -(1 + \frac{3g}{r}).$$

So,

$$-\frac{\partial(\theta + m\angle AOC - \pi)}{\partial \theta} < 0.$$

This means that as we decrease $m\angle GOA$, the $m\angle GOA + m\angle BOA - \pi$ decreases. So,

$$m\angle GOA + m\angle COA < \pi.$$

Therefore, A and C must be on the same side of the Euler line. This means that θ is made by $\angle GOB$, and

$$\angle GOB > \cos^{-1}\left(\frac{3g+r}{2r}\right),$$

otherwise C and B would be on the same side of the Euler line (and C cannot be on both sides of the Euler line). This proves that θ_L is the lower bound for θ when $g < \frac{r}{3}$.

Combining the upper and lower bounds we get,

$$\cos^{-1}\left(\frac{3g+r}{2r}\right) \leq \theta \leq \cos^{-1}\left(\frac{3g-r}{2r}\right).$$

□

Note that if we consider the $g = 0$ case, we have defined $\theta = \frac{\pi}{2}$ which falls between the bounds.

Proposition 16. *If $g = \frac{r}{3}$, then $0 < \theta \leq \frac{\pi}{2}$.*

Proof. To find the upper bound for θ when $g = \frac{r}{3}$ we again look at the isosceles triangle with vertex B on the same side of O as G . We know that in this case, O and M_b are the same point, and that $m\angle AOG = \frac{\pi}{2}$. We also know that A , O , and C are collinear, so if we increase $\angle AOG$ we necessarily decrease $\angle GOC$, and the two angles must sum to π . The right triangle case is the reason we had to add the requirement that θ be the smaller of the two angles if we have two “one-vertex sides.” In this case, the largest value that the smaller angle can take is $\frac{\pi}{2}$. When we go to look at the isosceles triangle with vertex C on the opposite side of O from G , we realize that we cannot make this triangle because two vertices would have to fall on on the Euler line, which is impossible. So if we keep B at the right angle, we can move vertex C arbitrarily close to the Euler line on the opposite side of the circle from B . This means that we can make $m\angle GOC$ arbitrarily close to π , so $m\angle GOA$ can be made arbitrarily close to 0. So when $g = \frac{r}{3}$

$$0 < \theta \leq \frac{\pi}{2}.$$

□

Proposition 17. *If $g > \frac{r}{3}$, then $\cos^{-1}\left(\frac{3g-r}{2r}\right) \leq \theta \leq \cos^{-1}\left(\frac{3g^2-r^2}{2rg}\right)$.*

Proof. Again we consider the isosceles triangle with vertex B on the Euler line on the same side of O that G is on. So,

$$m\angle GOA = \theta_R = \cos^{-1}\left(\frac{3g-r}{2r}\right).$$

As before we have $\angle GOA \cong \angle AOB$. Also, $\angle GOA \cong \angle GOC$, and both measure θ , so we can think of $\angle GOA$ as θ . Following the same procedure for the isosceles triangle with $\theta = \theta_R$ (from the $g < \frac{r}{3}$ case), we see,

$$\frac{\partial(\theta - m\angle AOB)}{\partial\theta} = 1 - \frac{3g}{r}.$$

But this time, since $g > \frac{r}{3}$,

$$\frac{\partial(\theta - m\angle AOB)}{\partial\theta} < 0.$$

This means that as we increase $m\angle GOA$, the difference $m\angle GOA - m\angle BOA$ decreases! So,

$$m\angle GOA - m\angle BOA < 0.$$

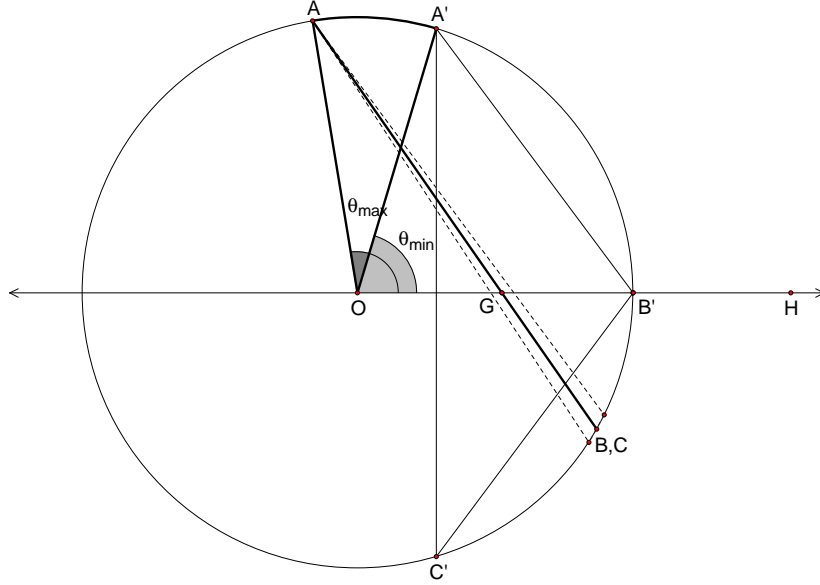


Figure 14: The bounds are created by an isosceles triangle and a line.

Therefore, A and B must be on opposite sides of the Euler line. This proves that θ_R is the lower bound for θ when $g > \frac{r}{3}$.

When $g > \frac{r}{3}$ there is no other isosceles triangle to look at. This is because $BM_b = \frac{3}{2}(r + g) > \frac{3}{2}(\frac{4r}{3}) = 2r$, so the midpoint of the side opposite B and therefore the side itself is not contained in the circumcircle. Even though this means that our usual means of finding the other bound is not possible, it gives us another possibility. Now we set out to find where the triangle disappears. To do this, we find the angle that forces M_a to land on the circumcircle (also forcing A , M_A , and G to be collinear). Therefore, we are interested in the case where $OA = r = OM_a$. This means $\triangle AOM_a$ is an isosceles triangle, and $\angle OAM_a \cong \angle AM_aO$. If we define $l = \overline{AG}$, then $AM_a = \frac{3l}{2}$. By the law of cosines, $r^2 = r^2 + \frac{9l^2}{4} - 3rl \cos \angle AM_aO$. After some simple algebra we discover,

$$2r(\cos \angle AMO) = \frac{3l}{2}.$$

By the Law of Cosines in $\triangle AOG$ we know $g^2 = r^2 + l^2 - 2rl \cos \angle OAM_a$. Substituting and solving for l^2 , we get $l^2 = 2(r^2 - g^2)$. We again use the law of cosines in $\triangle AOG$ and get $l^2 = r^2 + g^2 - 2rg \cos \theta$. Substituting for l^2 and solving for θ we get

$$\theta = \cos^{-1}\left(\frac{3g^2 - r^2}{2rg}\right).$$

Thus,

$$\cos^{-1}\left(\frac{3g - r}{2r}\right) \leq \theta \leq \cos^{-1}\left(\frac{3g^2 - r^2}{2rg}\right).$$

□

The nice part about the last upper bound for θ is that it is the inverse function to our upper bound for g . With that in mind, here is a cross-section of our space for $r = 1$:

As you see in the picture, we have built in a flipping effect that happens when g crosses the $\frac{r}{3}$ line. This stems directly from our definition of θ as the angle formed to the one-vertex side. It is a problem we can live with and will deal with shortly, but we do so with the belief that our definition of θ is the

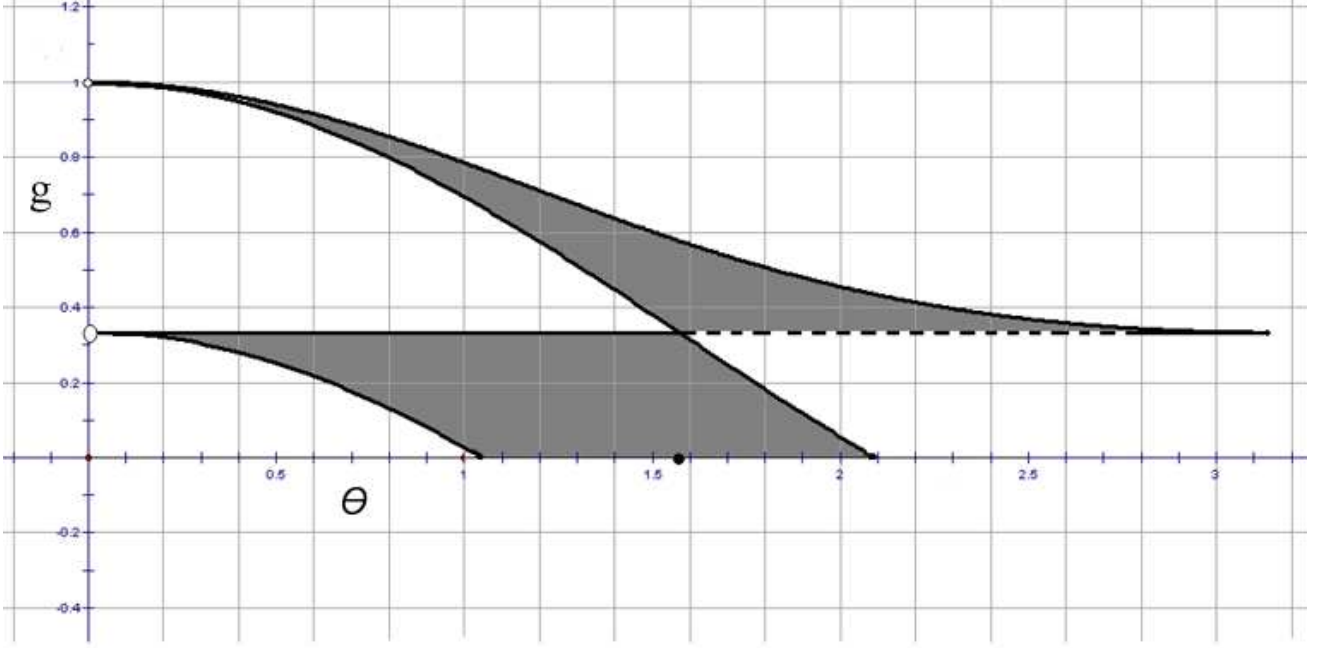


Figure 15: A cross-section of the space when $r = 1$.

most natural way to eliminate overcounting. First, we must quickly return to the situation where we fix θ and allow g to vary. If we do this and allow g to cross the $\frac{r}{3}$ threshold, it is not immediately clear what happens. Upon reflection of the $g = \frac{r}{3}$ case, we realize that $(\theta, g, r) \cong (\pi - \theta, g, r)$.

Now that we know where all of our triangles are and what triangle space looks like, it is natural to define a metric to determine how close two triangles are to being congruent. We would like the metric to have two additional properties:

- i) Account for the fact that all triangles with $g = 0$ and the same r are congruent, and
- ii) Account for the flipping effect when g crosses $\frac{r}{3}$.

First, we will account for the flipping by defining a new function,

$$\bar{\theta} = \begin{cases} \theta_i & g_i \leq \frac{r_i}{3} \\ \pi - \theta_i & g_i > \frac{r_i}{3} \end{cases} .$$

Let s and t be triangles such that $s = (\theta_1, g_1, r_1)$ and $t = (\theta_2, g_2, r_2)$. Define

$$D(s, t) = \sqrt{(g_1 \bar{\theta}_1 - g_2 \bar{\theta}_2)^2 + (g_1 - g_2)^2 + (r_1 - r_2)^2} .$$

Attaching the g 's onto the $\bar{\theta}$ terms accounts for how alike triangles with small g values are. It also leads to two nice propositions dealing with similar triangles, but first we will show a result about triangles who have the same r and g values.

Definition 8. A gr -family of triangles is a family of triangles with the same values of g and r .

Proposition 18. If triangles s and t are in the same gr -family, with $s = (\theta_1, g, r)$ and $t = (\theta_2, g, r)$, then $D(s, t) = g|\bar{\theta}_1 - \bar{\theta}_2|$.

Proof.

$$D(s, t) = \sqrt{(g\bar{\theta}_1 - g\bar{\theta}_2)^2 + (g - g)^2 + (r - r)^2} = \sqrt{g^2(\bar{\theta}_1 - \bar{\theta}_2)^2} = g|\bar{\theta}_1 - \bar{\theta}_2| .$$

□

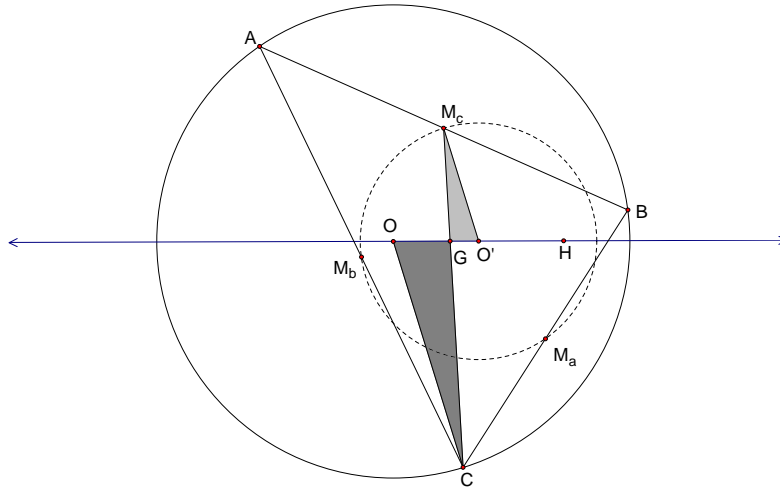


Figure 18: gr -family: the nine-point circle.

6 The Symmedian Point

Having examined the loci of certain well-known points in our construction, we now turn our attention to a more obscure triangle center—the symmedian point. In the collection of triangle centers, few could be considered well-known to geometers and even fewer to the average geometry student. Occasionally, however, centers which should be a part of mathematicians’ base knowledge disappear from the contemporary consciousness. The symmedian point is one such center. Well explored many years ago, the symmedian point has a plethora of useful and fascinating properties. In his work *19th and 20th Century Euclidean Geometry*, Ross Honsberger calls it “one of the jewels of modern geometry” [Honsberger, pg. 53]. In order to begin our brief study of this geometric gem, we first need to understand some established definitions and theorems. Thus, let us define the concept of isogonal conjugates.

Definition 10. Let $\angle A$ be given. Then, the isogonal conjugate of \overline{AP} , where P is any point in the plane, is the reflection of \overline{AP} over the angle bisector of $\angle A$. \overline{AP} and its reflection are called isogonal conjugate lines or simply isogonal conjugates.

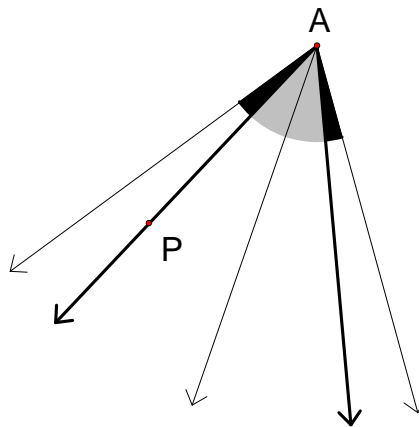


Figure 19: Isogonal conjugate lines.

In Figure 19 above, the middle line is the angle bisector of $\angle A$, and the two thick black lines are

isogonal conjugates. Moreover, as we can see above, one direct consequence of the definition of an isogonal conjugate is that the gray angles are congruent, and the black angles are congruent.

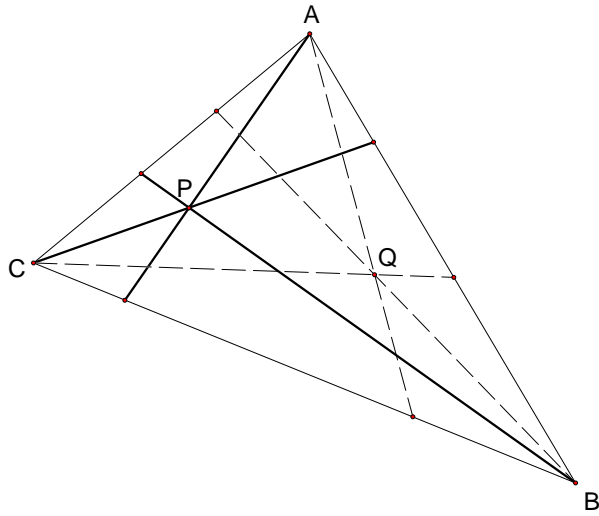


Figure 20: Isogonal conjugate points.

Since isogonal conjugates inherently involve angles, one question which naturally arises is how isogonal conjugates relate to triangles. As Theorem 2 below states, they have at least one fascinating property.

Theorem 2. *Let P be a point in the plane, and let $\triangle ABC$ be given. Then the lines isogonal to \overline{AP} , \overline{BP} , and \overline{CP} , meet at a point Q . Points P and Q are called isogonal conjugate points, or isogonal conjugates. [Honsberger, pg. 53]*

A proof of this theorem can be found in Ross Honsberger's work, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*. We will omit it here. However, Theorem 2 is an important part of this section since it states that isogonal conjugate points, like isogonal conjugate lines, occur in pairs. Surprisingly, O and H , the circumcenter and orthocenter of a triangle, are one such pair. Although this fact is well-known, our proof of it is included below. Before we begin, however, we must state a property of isogonal conjugates which we will use in the proof.

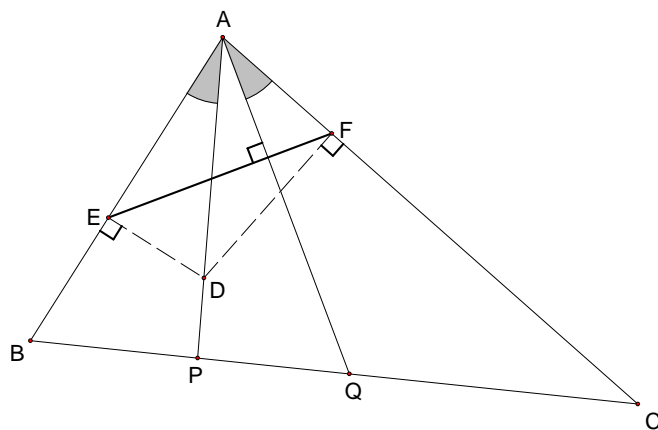


Figure 21: Proposition 24.

Proposition 24. Let \overline{AP} and \overline{AQ} be isogonal conjugate lines through vertex A , and let D be a point on \overline{AP} . Then the line \overline{EF} connecting the feet of the perpendiculars from D to \overline{AB} and \overline{AC} is perpendicular to \overline{AQ} . [Honsberger, pgs. 64–65]

A proof of Proposition 24 can be found in Honsberger. We will omit it here and move directly to our proof that the circumcenter and the orthocenter are isogonal conjugates.

Proposition 25. The circumcenter and the orthocenter of a triangle are isogonal conjugates.

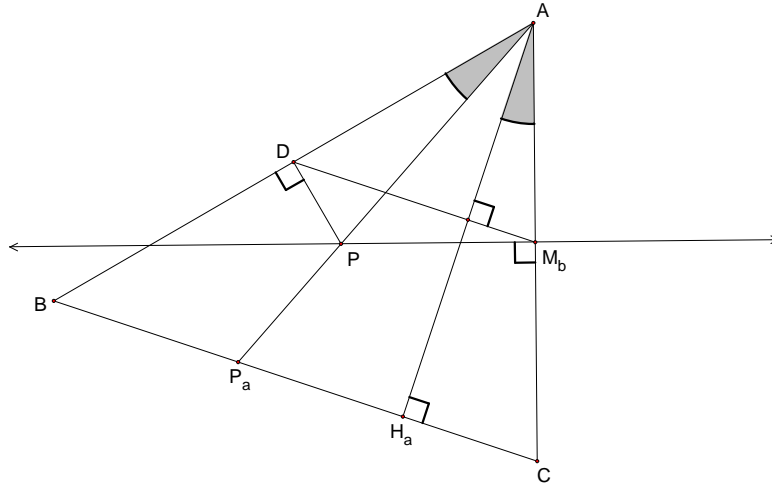


Figure 22: Isogonal conjugates: the circumcenter and the orthocenter.

Proof. Let $\triangle ABC$ be given. Construct $\overline{AH_a}$, the altitude from vertex A to \overline{BC} . Construct the isogonal conjugate of $\overline{AH_a}$. It will meet \overline{BC} at a point P_a . Now, construct the perpendicular bisector of \overline{AC} . It will meet $\overline{AP_a}$ at a point P . Drop a perpendicular from P to \overline{AB} , and call the intersection of the two D . Construct $\overline{DM_b}$, where M_b is the midpoint of \overline{AC} . By Proposition 24, $\overline{DM_b}$ must be perpendicular to $\overline{AH_a}$. However, by construction, $\overline{AH_a}$ is perpendicular to \overline{BC} . Thus, $\overline{DM_b} \parallel \overline{BC}$. By AAA similarity, $\triangle ADM_b \sim \triangle ABC$. So, $\frac{AD}{AB} = \frac{AM_b}{AC} = \frac{1}{2}$. Therefore, D is the midpoint of \overline{AB} , and P must be the circumcenter of $\triangle ABC$. Thus, P lies on the isogonal conjugate to the altitude from vertex A . By a similar argument, P will also lie on the isogonal conjugate to the altitude from vertices B and C . Therefore, the circumcenter and the orthocenter of $\triangle ABC$ are isogonal conjugates. \square

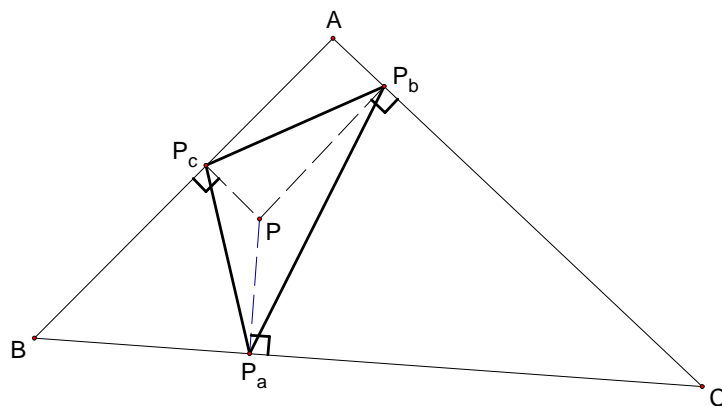


Figure 23: A pedal triangle.

Returning to more general isogonal conjugates, another known and interesting property which we will reference later relates the circumcircles of the pedal triangles of two isogonal conjugate points. Before we state and prove this property, however, we need to define the concepts of a pedal triangle and a cyclic quadrilateral, and prove a useful fact about cyclic quadrilaterals.

Definition 11. Let P be a point inside a given triangle $\triangle ABC$. Construct the perpendiculars from P to \overline{AB} , \overline{BC} , and \overline{AC} . Label the points of intersection P_c , P_a , and P_b , respectively. Then, $\triangle P_aP_bP_c$ is the pedal triangle of P with respect to $\triangle ABC$.

The next definition and lemma deal with certain types of quadrilaterals.

Definition 12. A quadrilateral $ABCD$ is called cyclic if all of its vertices lie on a circle.

Lemma 1. Let $ABCD$ be a quadrilateral with right angles at vertices A and C . Then, quadrilateral $ABCD$ is cyclic.

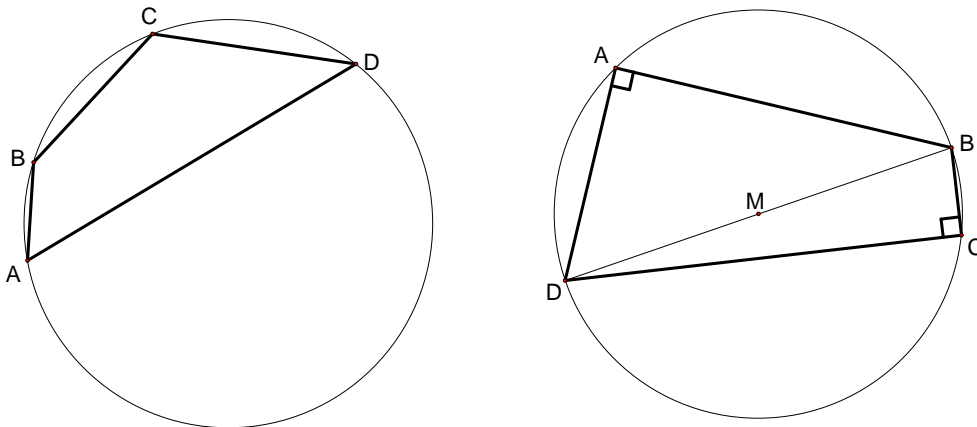


Figure 24: A generic cyclic quadrilateral and a cyclic quadrilateral with right angles.

Proof. Connect B to D . Then, the midpoint M of \overline{BD} will be the circumcenter of $\triangle ABD$ and of $\triangle CBD$. Thus, $\overline{MA} \cong \overline{MD} \cong \overline{MB} \cong \overline{MC}$. Thus, $ABCD$ lies on the circle centered at M , and $ABCD$ is cyclic. \square

Now, let us state and prove the property of isogonal conjugates which relates the circumcircles of the pedal triangles of two isogonal conjugate points. Note that although this fact is known, the following proof is our own.

Proposition 26. Let triangle $\triangle ABC$ be given with isogonal conjugate points P and Q . Then, the midpoint O of the segment \overline{PQ} is the circumcenter of the pedal triangles of both P and Q . Moreover, both pedal triangles share the same circumcircle. [Honsberger, pgs. 67–69]

Proof. The following proof utilizes Figure 25.

Let P and Q be isogonal conjugate points, and let O be the midpoint of segment \overline{PQ} . Let M be the midpoint of segment \overline{AP} , and let triangle $\triangle P_aP_bP_c$ be the pedal triangle determined by the point P . Let I be the intersection of segment $\overline{P_bP_c}$ with \overline{OM} , and let J be the intersection of segment $\overline{P_bP_c}$ with \overline{AQ} .

We first show that $\overline{OM} \parallel \overline{AQ}$. By construction, $MP = \frac{1}{2}AP$ and $OP = \frac{1}{2}PQ$. Since $\angle MPO$ is common, it follows that $\triangle MPO \sim \triangle APQ$. Thus, $\angle PMO \sim \angle PAQ$, which indicates that $\overline{OM} \parallel \overline{AQ}$.

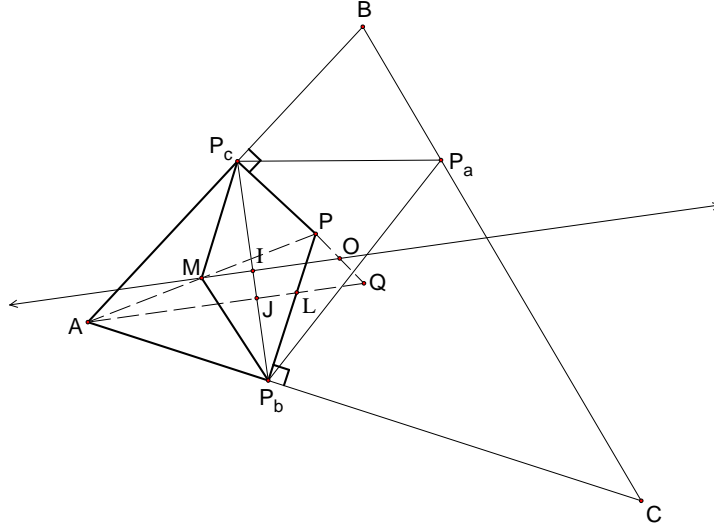


Figure 25: Proof that the pedal triangles of points P and Q share the same circumcenter.

Now consider triangles $\triangle AP_cP$ and $\triangle AP_bL$, where L is the intersection of $\overline{P_bP}$ with \overline{AQ} . By construction, $m\angle PP_cA = \frac{\pi}{2} = m\angle LP_bA$. Moreover, since P and Q are isogonal conjugates, $\angle P_cAP \cong \angle P_bAL$. By angle subtraction, $\angle P_cPA \cong \angle P_bLA$. Thus, by AAA Similarity, $\triangle AP_cP \sim \triangle AP_bL$.

Now, since $m\angle LP_bA = \frac{\pi}{2} = m\angle PP_cA$, quadrilateral AP_bPP_c is cyclic by Lemma 1. Thus, $\angle P_cAP \cong \angle P_cP_bP$. Since $m\angle P_cAP + m\angle P_cPA = \angle P_cP_bP + m\angle P_bLA = \frac{\pi}{2}$, it follows that $m\angle P_bJL = \frac{\pi}{2}$. This implies that $m\angle P_cIO = \frac{\pi}{2}$ since $\overline{OM} \parallel \overline{AQ}$.

Now, since quadrilateral AP_bPP_c is cyclic, the points A , P_b , P , and P_c lie on a circle centered at M , the midpoint of \overline{AP} . Thus, $\overline{MP_c} \cong \overline{MP_b}$. Moreover, since segment \overline{MI} is common, $\triangle MP_cI$ and $\triangle MP_bI$ both have right angles, congruent hypotenuses, and one congruent leg. They are consequently congruent. This then indicates that $\overline{IP_c} \cong \overline{IP_b}$, making I the midpoint of $\overline{P_bP_c}$.

Therefore, \overline{OM} is the perpendicular bisector of $\overline{P_bP_c}$. A similar argument shows that O also lies on the perpendicular bisectors to segments $\overline{P_aP_b}$ and $\overline{P_aP_c}$, making O the circumcenter of $\triangle P_aP_bP_c$.

Repeat the argument above for Q . Then, the midpoint O of segment \overline{PQ} is the circumcenter of the pedal triangles constructed from P and Q .

To show the two pedal triangles share not only the same circumcenter, but also the same circumcircle, consider the following argument, which utilizes Figure 26 below.

Let points P and Q be isogonal conjugates, and let O be the midpoint of segment \overline{PQ} . Drop perpendiculars from P , O , and Q to \overline{AC} , and let P_b , O_b , and Q_b be their intersections with \overline{AC} , respectively. Since $m\angle PP_bQ_b = m\angle OO_bQ_b = m\angle QQ_bO_b = \frac{\pi}{2}$, it follows that $\overline{PP_b} \parallel \overline{OO_b} \parallel \overline{QQ_b}$. Construct $\overline{PQ_b}$ and let N be the intersection of $\overline{PQ_b}$ and $\overline{OO_b}$. Then, $\angle PNO \cong \angle PQ_bQ$ and $\angle PON \cong \angle PQ_bQ$. Since $\angle NPO$ is common, $\triangle PNO \sim \triangle PQ_bQ$. Thus, $\frac{PO}{PQ} = \frac{PN}{PQ_b} = \frac{1}{2}$, and N is the midpoint of segment $\overline{PQ_b}$. By a similar argument, $\triangle Q_bO_bN \sim \triangle Q_bP_bP$. Thus, $\frac{PN}{PQ_b} = \frac{P_bO_b}{P_bQ_b} = \frac{1}{2}$. This implies that O_b is the midpoint of segment $\overline{P_bQ_b}$. Thus, since $\overline{P_bO_b} \cong \overline{P_bQ_b}$, $\angle P_bO_bO \cong \angle Q_bO_bO$, and $\overline{OO_b}$ is common, by SAS congruence, $\triangle OO_bP_b \cong \triangle OO_bQ_b$. This indicates that $\overline{OP_b} \cong \overline{OQ_b}$. However, $\overline{OP_b}$ is the radius of the pedal triangle of P , and $\overline{OQ_b}$ is the radius of the pedal triangle of Q . Therefore, the pedal triangles of P and Q share the same circumcircle. \square

Now that we have a basic understanding of isogonal conjugates and some of their properties, we are ready to define the symmedian point.

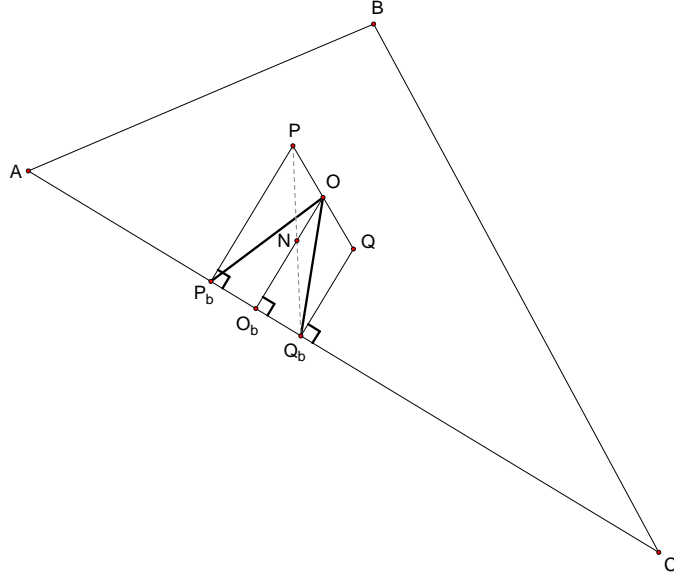


Figure 26: Proof that the pedal triangles of points P and Q share the same circumcircle.

Definition 13. The symmedian point K is the isogonal conjugate of the centroid.

Interesting properties of the symmedian include such statements as Proposition 27 below.

Proposition 27. *The symmedian point is the centroid of its own pedal triangle. [Honsberger, pgs. 72–73]*

This proposition is well known. A proof of it can be found in Honsberger. Moreover, utilizing it and Proposition 26, we can immediately prove the following corollary.

Corollary 2. *The centroid of a $\triangle ABC$ always lies on the Euler line of the pedal triangle of its symmedian point K .*

Proof. Let $\triangle ABC$ be given. Since the centroid and the circumcenter of a triangle determine its Euler line, it follows from Propositions 26 and 27 that \overline{GK} , where G is the centroid of $\triangle ABC$ and K is its symmedian point, is the Euler line of the pedal triangle of K . \square

Corollary 2 leads to an interesting observation. The locus of symmedian points in a gr -family of triangles form either an entire circle or an arc of a circle centered on the Euler line. Let us state this more formally.

Theorem 3. *Let Ω be the family of triangles produced by fixing a circumcenter O , a centroid G , a circumradius r , and varying $\angle GOA$ from 0 to 2π . Let g denote the distance between O and G . Let $E \subseteq [0, 2\pi]$ such that for all $\phi \in E$, there exists a triangle $\triangle ABC_\phi \in \Omega$ with $m\angle GOA = \phi$. Let K_ϕ be the symmedian point of $\triangle ABC_\phi$.*

Let K_n be the point on the ray \overrightarrow{OG} such that $OK_n = \frac{2gr^2}{r^2 - g^2}$. Then, for all $\phi \in E$, K_ϕ lies on the circle with radius $\frac{2g^2r}{r^2 - g^2}$ centered at K_n . We call this circle the Carleton Circle and its center K_n the Knights' Point.

If $g < \frac{r}{3}$, every point on the Carleton Circle is the symmedian point of a triangle in Ω . If $g = \frac{r}{3}$, then every point except $(r, 0)$ is the symmedian point of a triangle in Ω . If $g > \frac{r}{3}$, then every point P on the Carleton Circle such that P is strictly contained in the interior of the disc enclosed by the circumcircle of Ω is the symmedian point of a triangle in Ω .

While in the specific case in which $g = \frac{r}{3}$ a geometric proof of Theorem 3 is apparent, in the general case, such a proof is not obvious. Thus, we will approach the general case from an analytic perspective. However, in order to more clearly understand precisely what Theorem 3 states, let us begin our exploration with the geometric proof of the case in which $g = \frac{r}{3}$.

Lemma 2. *Let Ω be a gr -family of triangles in which $g = \frac{r}{3}$. Let K_n lie on segment \overline{OH} such that $OK_n = 3K_nH$. Let the circle centered at K_n with radius K_nH be called the Carleton Circle. Then, for every $\triangle ABC_\phi \in \Omega$, K_ϕ , the symmedian point of $\triangle ABC_\phi$, lies on the Carleton Circle. Moreover, every point except H on the Carleton Circle is the symmedian point of a triangle in the given Ω .*

Note that Lemma 2's geometric description of the location of K_n and the radius of the Carleton Circle correspond to the analytic description in Theorem 3. This is because by Proposition 7 and Corollary 1, when $g = \frac{r}{3}$, H will coincide with one vertex of $\triangle ABC_\phi$, making \overline{OH} a radius of the circumcircle. Thus, K_n will lie the following distance along the Euler line of $\triangle ABC$: $\frac{2gr^2}{r^2-g^2} = \frac{2g(3g)^2}{(3g)^2-g^2} = \frac{18g^3}{8g^2} = \frac{9g}{4} = \frac{3r}{4} = \frac{3}{4}OH$. Moreover, the radius of the Carleton Circle will be $\frac{2g^2r}{r^2-g^2} = \frac{2g^2(3g)}{(3g)^2-g^2} = \frac{6g^3}{8g^2} = \frac{3g}{4} = \frac{r}{4} = \frac{1}{4}OH$.

Now, to prove Lemma 2, we will use the following proposition.

Proposition 28. *Let $\triangle ABC$ be a right triangle with the right angle at vertex B . Then, the symmedian point K is the midpoint of the symmedian from vertex B . [Honsberger, pgs. 59–60]*

This fact is commonly known; a proof of it can be found in Honsberger. Let us now prove Lemma 2.

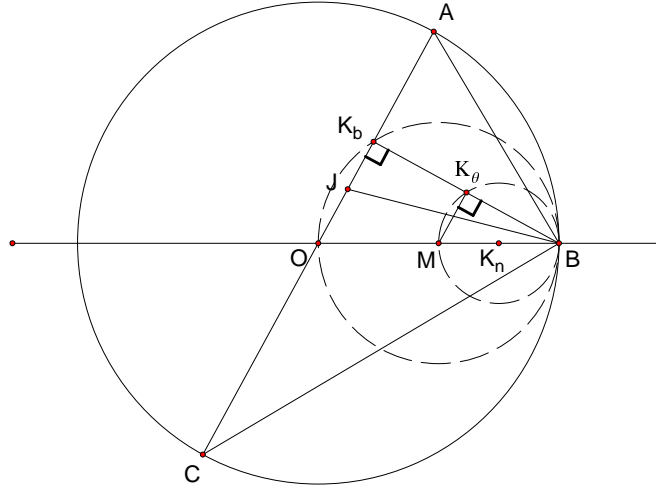


Figure 27: Proof that the symmedian point lies on a circle when $\triangle ABC$ is a right triangle.

Proof. By Corollary 1 and Proposition 6, one vertex of $\triangle ABC_\phi$ must coincide with H . Without loss of generality, assume that B is that vertex. Construct the symmedian from B , and let K_b be the intersection of that symmedian with \overline{AC} . We will first show that K_b lies on the circle centered at M , the midpoint of segment \overline{OH} .

Construct the angle bisector of $\angle ABC$ and label it \overline{JB} , where J is its intersection with \overline{AC} . Now, since the symmedian is the isogonal conjugate of the median, $\angle OBJ \cong \angle JBK_b$ and $\angle K_bBA \cong \angle OBC$. Since O is the circumcenter of $\triangle ABC$, it follows that $\triangle OBC$ is isosceles with $\overline{OB} \cong \overline{OC}$.

Thus, $\angle ACB \cong \angle OBC$. Now, $m\angle ABC = \frac{\pi}{2}$. This implies that $m\angle ACB + m\angle CAB = \frac{\pi}{2}$ which in turn implies that $m\angle K_bBA + m\angle CAB = \frac{\pi}{2}$. Thus, $m\angle AK_bB = \frac{\pi}{2}$, and K_b lies on the circle centered at M with radius MO . Note that in this shows that K_b is the altitude from vertex B , and that the circle it lies on is the circumcircle of the pedal triangles of O and H .

Now, by Proposition 28, K_ϕ is the midpoint of $\overline{K_bB}$. Construct $\overline{K_\phi M}$. By SAS similarity, $\triangle K_\phi BM \sim \triangle K_bBO$. Thus, $m\angle MK_\phi B = m\angle OK_bB = \frac{\pi}{2}$. Therefore, K_ϕ lies on the circle centered at the midpoint K_n of \overline{MB} , $\frac{3}{4}$ of the way from O to H . The radius of the circle will be $\frac{1}{2}MB = \frac{1}{4}OH$.

To show that every point except B on the Carleton Circle is a symmedian point of a triangle in the given gr -family, let J be a point on the Carleton Circle. Since H is on the circumcircle, by Proposition 6, it must coincide with one vertex of every triangle in the given gr -family. Without loss of generality, let B be that vertex. Construct \overline{BJ} . Then, extend \overline{BJ} past J to a point J_b such that $\overline{JJ_b} \cong \overline{BJ}$. By the argument above, J_b is the foot of the altitude from B . Construct the line perpendicular to $\overline{J_bB}$ through J_b . This line will intersect the circumcircle at points A and C . By construction, $\triangle ABC_\phi$ has symmedian point J . Thus, every point except B on the Carleton Circle centered at K_n is a symmedian point of a triangle in the given gr -family.¹ \square

To show the general case of Theorem 3, we will first think about placing the gr -family of triangles in a coordinate system. We will find the x - and y -coordinates of the Knights' Point K_n and of the symmedian point K_ϕ of any triangle in that family. Then, we will use the distance formula to find the distance between these two points. If that distance is not dependent on ϕ , then, since K_n is fixed in a gr -family, K_ϕ will always lie on a circle centered at K_n .

The proof below will make use to two propositions, both of which are stated below. We will not include proofs of either here.

Proposition 29. *Let vectors \vec{a} and \vec{b} be given. Then, $\vec{c} = |\vec{b}|\vec{a} + |\vec{a}|\vec{b}$ bisects the angle created by vectors \vec{a} and \vec{b} . [Stewart, pg. 850]*

Proposition 30. *Let triangle $\triangle ABC$ with altitude $\overline{AH_a}$ be given. Then, the symmedian point K of $\triangle ABC$ lies on the line connecting the midpoint M_h of $\overline{AH_a}$ to the midpoint M_a of \overline{BC} . [Honsberger, pgs. 65-67]*

Now, let us prove Theorem 3.

Proof. Let Ω be the gr -family of triangles with circumcenter O , centroid G , and circumradius r given. As we have previously done, let g denote the distance between the circumcenter and the centroid. Let ϕ denote $m\angle GOA$ where A is the chosen vertex in the triangle construction. Choose O to be the origin of the coordinate system and the Euler line \overline{OG} to be the x -axis. We will first find expressions for the coordinates of the vertices A , B , and C of any triangle in the gr -family. Our method of finding them will algebraically mimic the geometric construction described earlier.

By construction, O has coordinates $(0, 0)$, G has coordinates $(g, 0)$, and H has coordinates $(3g, 0)$. Since A lies on the circle with radius r , and since angle ϕ is the angle between \overrightarrow{OA} and

¹The reason that B cannot be a symmedian point is a result of the fact that the symmedian point K of a $\triangle ABC$ can never fall on the circumcircle of $\triangle ABC$. Suppose that K does lie on the circumcircle. Then, either K is coincident with one vertex, say vertex A , or K is not coincident with any vertex of $\triangle ABC$. Suppose K is coincident with vertex A . Then, \overline{AB} and \overline{AC} are the symmedian lines from vertices B and C . Consequently, \overline{BC} must be the median from both vertices B and C . However, that would force C to be the midpoint of \overline{AC} and B to be the midpoint of \overline{AB} . This would imply that A , B , and C are coincident. This is a contradiction. Now, suppose that K is not coincident with any vertex of $\triangle ABC$. Then, K lies outside $\triangle ABC$. However, that would force the median of at least one vertex to lie outside of $\triangle ABC$. This is a contradiction. Thus, K must always be strictly inside the circumcircle of $\triangle ABC$.

the x -axis, it follows that A has coordinates $(r \cos \phi, r \sin \phi)$. Thus, we can find the equation of the line \overline{AG} by computing its slope and its y -intercept.

The slope of the \overline{AG} is

$$\frac{r \sin \phi - 0}{r \cos \phi - g} = \frac{r \sin \phi}{r \cos \phi - g}.$$

To find the y -intercept b , we use the point $(0, g)$ and the slope:

$$\begin{aligned} 0 &= g \frac{r \sin \phi}{r \cos \phi - g} + b \\ b &= -g \frac{r \sin \phi}{r \cos \phi - g} \end{aligned}$$

Thus, the equation of \overline{AG} is $y = \frac{r \sin \phi}{r \cos \phi - g}x - g \frac{r \sin \phi}{r \cos \phi - g}$ or equivalently $y = \frac{r \sin \phi(x-g)}{r \cos \phi - g}$.

Next, we find the coordinates of the midpoint M_a of \overline{BC} . Since the y -coordinate of G is 0, and since M_a lies on the ray \overrightarrow{AG} such that $AM_a = \frac{3}{2}AG$, the y -coordinate of M_a must be $\frac{-r \sin \phi}{2}$. We solve for the x -coordinate using the equation of \overline{AG} :

$$\begin{aligned} \frac{r \sin \phi}{r \cos \phi - g}x - g \frac{r \sin \phi}{r \cos \phi - g} &= \frac{-r \sin \phi}{2} \\ \frac{r \sin \phi}{r \cos \phi - g}x &= \frac{-r \sin \phi}{2} + g \frac{r \sin \phi}{r \cos \phi - g} \\ \frac{r \sin \phi}{r \cos \phi - g}x &= \frac{-r \sin \phi(r \cos \phi - g) + 2gr \sin \phi}{2(r \cos \phi - g)} \\ x &= \left(\frac{-r \sin \phi(r \cos \phi - g) + 2gr \sin \phi}{2(r \cos \phi - g)} \right) \left(\frac{r \cos \phi - g}{r \sin \phi} \right) \\ x &= \frac{g - r \cos \phi + 2g}{2} \\ x &= \frac{3g - r \cos \phi}{2}. \end{aligned}$$

Thus, the coordinates of M_a are $(\frac{3g - r \cos \phi}{2}, \frac{-r \sin \phi}{2})$. Let us now find the equation of \overline{AH} .

The slope of the \overline{AH} is

$$\frac{r \sin \phi - 0}{r \cos \phi - 3g} = \frac{r \sin \phi}{r \cos \phi - 3g}.$$

Thus, we now only need its y -intercept d , which we find using the point $(0, 3g)$ and the slope:

$$\begin{aligned} 0 &= 3g \frac{r \sin \phi}{r \cos \phi - 3g} + d \\ d &= -3g \frac{r \sin \phi}{r \cos \phi - 3g} \end{aligned}$$

The equation of \overline{AH} is $y = \frac{r \sin \phi}{r \cos \phi - 3g}x - 3g \frac{r \sin \phi}{r \cos \phi - 3g}$ or equivalently $y = \frac{r \sin \phi(x - 3g)}{r \cos \phi - 3g}$.

To find \overline{BC} , we use the fact that \overline{BC} is perpendicular to \overline{AH} . From this, it follows that the slope of \overline{BC} is the negative reciprocal of the slope of \overline{AH} . Thus, the slope of \overline{BC} must be $\frac{r \cos \phi - 3g}{r \sin \phi}$.

We now find the y -intercept i of \overline{BC} using its slope and the coordinates of M_a :

$$\begin{aligned}
\left(\frac{3g - r \cos \phi}{r \sin \phi}\right)\left(\frac{3g - r \cos \phi}{2}\right) + i &= \frac{-r \sin \phi}{2} \\
\frac{(3g - r \cos \phi)^2}{2r \sin \phi} + i &= \frac{-r \sin \phi}{2} \\
i &= \frac{-r \sin \phi}{2} - \frac{(3g - r \cos \phi)^2}{2r \sin \phi} \\
i &= \frac{-r^2 \sin^2 \phi - (3g - r \cos \phi)^2}{2r \sin \phi}.
\end{aligned}$$

Thus, the equation of \overline{BC} is as follows:

$$y = \frac{3g - r \cos \phi}{r \sin \phi}x + \frac{-r^2 \sin^2 \phi - (3g - r \cos \phi)^2}{2r \sin \phi}.$$

To find the coordinates of B and C , we solve the system of equations

$$y = \frac{3g - r \cos \phi}{r \sin \phi}x + \frac{-r^2 \sin^2 \phi - (3g - r \cos \phi)^2}{2r \sin \phi}.$$

and

$$y^2 + x^2 = r^2.$$

Using Mathematica to solve this complicated system, we see that the two solutions are

$$\begin{aligned}
x_B &= \frac{54g^3 + 12gr^2 - 2r(27g^2 + r^2) \cos \phi + 6gr^2 \cos 2\phi}{4(9g^2 + r^2 - 6gr \cos \phi)} \\
&\quad + \frac{r\sqrt{3} \csc \phi (\cos 2\phi - 1) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{4(9g^2 + r^2 - 6gr \cos \phi)}.
\end{aligned}$$

and

$$\begin{aligned}
x_C &= \frac{54g^3 + 12gr^2 - 2r(27g^2 + r^2) \cos \phi + 6gr^2 \cos 2\phi}{4(9g^2 + r^2 - 6gr \cos \phi)} \\
&\quad - \frac{r\sqrt{3} \csc \phi (\cos 2\phi - 1) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{4(9g^2 + r^2 - 6gr \cos \phi)}.
\end{aligned}$$

To find the y -coordinates of B and C , we plug their x -values into the equation of \overline{BC} . Doing so, we find that

$$\begin{aligned}
y_B &= -\sin \phi \left(\frac{9g^2r^2 + r^4 - 6gr^3 \cos \phi}{2r(9g^2 + r^2 - 6gr \cos \phi)} \right. \\
&\quad \left. + \frac{r\sqrt{3} \csc \phi (3g - r \cos \phi) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{2r(9g^2 + r^2 - 6gr \cos \phi)} \right) \\
&= -\sin \phi \left(\frac{r^2(9g^2 + r^2 - 6gr \cos \phi)}{2r(9g^2 + r^2 - 6gr \cos \phi)} \right. \\
&\quad \left. + \frac{\sqrt{3} \csc \phi (3g - r \cos \phi) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{2(9g^2 + r^2 - 6gr \cos \phi)} \right) \\
&= -\sin \phi \left(\frac{r}{2} + \frac{\sqrt{3} \csc \phi (3g - r \cos \phi) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{2(9g^2 + r^2 - 6gr \cos \phi)} \right).
\end{aligned}$$

and

$$\begin{aligned}
y_C &= -\sin \phi \left(\frac{9g^2r^2 + r^4 - 6gr^3 \cos \phi}{2r(9g^2 + r^2 - 6gr \cos \phi)} \right. \\
&\quad \left. - \frac{r\sqrt{3} \csc \phi (3g - r \cos \phi) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{2r(9g^2 + r^2 - 6gr \cos \phi)} \right) \\
&= -\sin \phi \left(\frac{r^2(9g^2 + r^2 - 6gr \cos \phi)}{2r(9g^2 + r^2 - 6gr \cos \phi)} \right. \\
&\quad \left. - \frac{\sqrt{3} \csc \phi (3g - r \cos \phi) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{2(9g^2 + r^2 - 6gr \cos \phi)} \right) \\
&= -\sin \phi \left(\frac{r}{2} - \frac{\sqrt{3} \csc \phi (3g - r \cos \phi) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{2(9g^2 + r^2 - 6gr \cos \phi)} \right).
\end{aligned}$$

Thus, the coordinates of A , B , and C are as follows:

$$A = (r \cos \phi, r \sin \phi).$$

$$\begin{aligned}
B &= \left(\frac{54g^3 + 12gr^2 - 2r(27g^2 + r^2) \cos \phi + 6gr^2 \cos 2\phi}{4(9g^2 + r^2 - 6gr \cos \phi)} \right. \\
&\quad \left. + \frac{r\sqrt{3} \csc \phi (\cos 2\phi - 1) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{4(9g^2 + r^2 - 6gr \cos \phi)}, \right. \\
&\quad \left. -\sin \phi \left(\frac{r}{2} + \frac{\sqrt{3} \csc \phi (3g - r \cos \phi) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{2(9g^2 + r^2 - 6gr \cos \phi)} \right) \right).
\end{aligned}$$

$$\begin{aligned}
C &= \left(\frac{54g^3 + 12gr^2 - 2r(27g^2 + r^2) \cos \phi + 6gr^2 \cos 2\phi}{4(9g^2 + r^2 - 6gr \cos \phi)} \right. \\
&\quad \left. - \frac{r\sqrt{3} \csc \phi (\cos 2\phi - 1) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{4(9g^2 + r^2 - 6gr \cos \phi)}, \right. \\
&\quad \left. -\sin \phi \left(\frac{r}{2} - \frac{\sqrt{3} \csc \phi (3g - r \cos \phi) \sqrt{-27g^4 + r^4 + 36g^3r \cos \phi - 4gr^3 \cos \phi - 6g^2r^2 \cos 2\phi}}{2(9g^2 + r^2 - 6gr \cos \phi)} \right) \right).
\end{aligned}$$

With the coordinates of the vertices of a generic triangle in Ω , we can now find the coordinates of the symmedian point.

We will begin by finding the equation of the symmedian line from vertex A . In order to do so, we need to bisect $\angle BAC$. However, unlike in non-analytic geometry, this is not a simple matter algebraically. We thus make use of Proposition 29 above.

Let us consider segments \overline{AB} and \overline{AC} to be vectors with their heads at B and C , respectively. To use Proposition 29 to bisect them, we first need to find their components. This is easily accomplished by subtracting the coordinates of their tails from the coordinates of their heads. We next find the magnitude of each vector. We will write neither the components nor the magnitudes of vectors \overline{AB} and \overline{AC} here as they are algebraically intensive. However, they are in the Mathematica document in the Appendix B .

Our next task is to multiply the components of \overline{AB} by the magnitude of \overline{AC} and to multiply the components of \overline{AC} by the magnitude of \overline{AB} . Adding these new scaled vectors by together, we find

by Proposition 29 a vector which bisects vectors \overrightarrow{AB} and \overrightarrow{AC} . Again, as this would simply be an exercise in precise copying, we have only included the resulting components in the Appendix B.

Now, to find the equation of the angle bisector of $\angle BAC$, we first find its slope—the y -coordinate over the x -coordinate of the angle bisector. Then, using this slope and the coordinates of vertex A , we can find the y -intercept. Thus, we have the equation of the angle bisector of $\angle BAC$.

As discussed previously, transformations such as reflections are significantly more difficult algebraically than they are geometrically. Thus, to find a point on the symmedian line from vertex A , we first find the equation of the line through G , ($(g, 0)$ in our coordinate system) perpendicular to the angle bisector of $\angle BAC$. This appears in Appendix B. Then, we find the x -coordinate of the intersection J of this new perpendicular line (which we will now refer to as \overline{GJ}) with the angle bisector. This value also appears in Appendix B.

Next, we find the point L on \overline{GJ} such that $LJ = GJ$. We do this by subtracting g from the x -value of J then adding that difference to the x -value of J . To find the y -value of L , we evaluate the equation of \overline{GJ} at the x -value we just computed. Thus, we now have the coordinates of L . L will be a point on the symmedian line from vertex A by the following geometric argument:

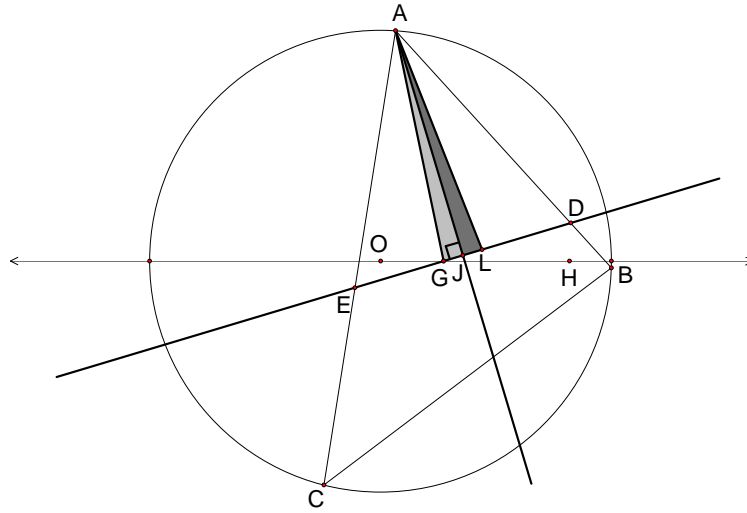


Figure 28: Constructing a point on the symmedian line from vertex A .

Let $\triangle ABC$ be given with centroid G . Construct the angle bisector of $\angle BAC$. Then, construct the line through G perpendicular to the angle bisector of $\angle BAC$. We have two cases:

Case 1. Suppose that \overline{AG} is not the angle bisector of $\angle BAC$. Then, the perpendicular line through G will meet the angle bisector at a point J . Now, construct the point L on \overline{GJ} such that $LJ = GJ$. Then, since $m\angle LJA = m\angle GJA = \frac{\pi}{2}$ and segment \overline{AJ} is common, it follows from SAS congruence that $\triangle AJG \cong \triangle AJL$. Thus, $\angle GAJ \cong \angle LAJ$, which implies that L is a point on the symmedian line from vertex A .

Case 2. Suppose that \overline{AG} is the angle bisector of $\angle BAC$. Then, the line through G perpendicular to \overline{AG} will intersect \overline{AG} at G . Then, G , J , and L will be coincident, and G will trivially be a point on the symmedian line from vertex A .

Now, using the coordinates of vertex L and vertex A , we can find the equation of the symmedian line from vertex A . Its equation simplifies nicely to the following.

$$y = \frac{(-2gr^2 + 3g^2x + r^2x - 2grx \cos \phi) \sin \phi}{(3g^2 + r^2) \cos \phi - gr(3 + \cos 2\phi)}.$$

Shortly after setting out to apply the method we employed to find the symmedian from A to find the symmedian from B , one realizes that given the coordinates of vertices B and C , the algebra is practically impossible to do by hand and takes a long time for even a program such as Mathematica to complete. Thus, we will use Proposition 30 above and find the equation of the line connecting the midpoint of side \overline{BC} to the midpoint of segment $\overline{AH_a}$, where H_a is the intersection of the altitude from A and \overline{BC} .

Thus, we first find where \overline{AH} intersects \overline{BC} . Recall that the equation of \overline{AH} is

$$y = \frac{r \sin \phi (x - 3g)}{r \cos \phi - 3g},$$

and that the equation of \overline{BC} is

$$y = \frac{3g - r \cos \phi}{r \sin \phi} x + \frac{-r^2 \sin^2 \phi - (3g - r \cos \phi)^2}{2r \sin \phi}.$$

Thus, we solve the system of equations above for x .

$$\begin{aligned} \frac{r \sin \phi (x - 3g)}{r \cos \phi - 3g} &= \frac{3g - r \cos \phi}{r \sin \phi} x + \frac{-r^2 \sin^2 \phi - (3g - r \cos \phi)^2}{2r \sin \phi} \\ -r^2 \sin^2 \phi (x - 3g) &= (3g - r \cos \phi)^2 x - \frac{r^2 \sin^2 \phi}{2} (3g - r \cos \phi) - \frac{(3g - r \cos \phi)^3}{2} \\ &\dots = \dots \\ x &= \frac{9g(3g^2 + r^2) - r(27g^2 + r^2) \cos \phi}{2(9g^2 + r^2 - 6gr \cos \phi)}. \end{aligned}$$

To find the y -coordinate of this point, we evaluate the equation of \overline{AH} at the given value of x above. Doing this, we find that the y -coordinate of H_A is $\frac{r(9g^2 - r^2) \sin \phi}{2(9g^2 + r^2 - 6gr \cos \phi)}$. We now find the midpoint of AH_a by adding the x -coordinates of A and H_a and dividing by 2.

$$\begin{aligned} &\frac{1}{2} \left(\frac{9g(3g^2 + r^2) - r(27g^2 + r^2) \cos \phi}{2(9g^2 + r^2 - 6gr \cos \phi)} + r \cos \phi \right) \\ &= \frac{27g^3 + 3gr^2 - 9g^2 r \cos \phi + r^3 \cos \phi - 6gr^2 \cos^2 \phi + 6gr^2 \sin^2 \phi}{4(9g^2 + r^2 - 6gr \cos \phi)}. \end{aligned}$$

We find its y -coordinate by plugging this value into the equation of \overline{AH} . Doing this, we find that the y -value of the midpoint of AH_a is as follows:

$$\frac{r(27g^2 + r^2 - 12gr \cos \phi) \sin \phi}{4(9g^2 + r^2 - 6gr \cos \phi)}.$$

Thus, we now have all the information we need to find the equation of the line $\overline{M_a H_a}$. We first find its slope.

$$\begin{aligned} &\frac{\frac{r(27g^2 + r^2 - 12gr \cos \phi) \sin \phi}{4(9g^2 + r^2 - 6gr \cos \phi)} - \left(-\frac{r \sin \phi}{2}\right)}{\frac{27g^3 + 3gr^2 - 9g^2 r \cos \phi + r^3 \cos \phi - 6gr^2 \cos^2 \phi + 6gr^2 \sin^2 \phi}{4(9g^2 + r^2 - 6gr \cos \phi)} - \left(\frac{3g - r \cos \phi}{2}\right)} \\ &= \frac{r(15g^2 + r^2 - 8gr \cos \phi) \sin \phi}{r(15g^2 + r^2) \cos \phi - g(9g^2 + 3r^2 + 4r^2 \cos(2\phi))}. \end{aligned}$$

Next we find j , the y -intercept of $\overline{M_a H_a}$, by using its slope and the point M_a with coordinates $(\frac{3g-r \cos \phi}{2}, \frac{-r \sin \phi}{2})$.

$$\begin{aligned} -\frac{1}{2}r \sin \phi &= \left(\frac{r(15g^2 + r^2 - 8gr \cos \phi) \sin \phi}{r(15g^2 + r^2) \cos \phi - g(9g^2 + 3r^2 + 4r^2 \cos(2\phi))} \right) \left(\frac{3g - r \cos \phi}{2} \right) + j \\ j &= -\frac{1}{2}r \sin \phi - \left(\frac{r(15g^2 + r^2 - 8gr \cos \phi) \sin \phi}{r(15g^2 + r^2) \cos \phi - g(9g^2 + 3r^2 + 4r^2 \cos(2\phi))} \right) \left(\frac{3g - r \cos \phi}{2} \right) \\ j &= \frac{2gr(9g^2 + r^2 - 6gr \cos \phi) \sin \phi}{-r(15g^2 + r^2) \cos \phi + g(9g^2 + 3r^2 + 4r^2 \cos 2\phi)}. \end{aligned}$$

Thus, we see that the equation of $\overline{M_a H_a}$, the line joining the midpoint of $\overline{AH_A}$ to the midpoint of \overline{BC} is as follows:

$$\begin{aligned} y &= \frac{r(15g^2 + r^2 - 8gr \cos \phi) \sin \phi}{r(15g^2 + r^2) \cos \phi - g(9g^2 + 3r^2 + 4r^2 \cos(2\phi))} x \\ &\quad + \frac{2gr(9g^2 + r^2 - 6gr \cos \phi) \sin \phi}{-r(15g^2 + r^2) \cos \phi + g(9g^2 + 3r^2 + 4r^2 \cos 2\phi)} \\ &= \frac{r(-18g^3 - 2gr^2 + 15g^2x + r^2x + 4gr(3g - 2x) \cos \phi) \sin \phi}{-9g^3 - 3gr^2 + 15g^2r \cos \phi + r^3 \cos \phi - 4gr^2 \cos^2 \phi + 4gr^2 \sin^2 \phi}. \end{aligned}$$

By Proposition 30, the symmedian point K_ϕ of $\triangle ABC_\phi$ lies on $\overline{M_a H_a}$. Since the symmedian point also lies on the symmedian line from vertex A , it follows that these two lines intersect at K_ϕ . To find the coordinates of K_ϕ , we solve the system of these two equations for x .

$$\begin{aligned} &\frac{(-2gr^2 + 3g^2x + r^2x - 2grx \cos \phi) \sin \phi}{(3g^2 + r^2) \cos \phi - gr(3 + \cos 2\phi)} \\ &= \frac{r(-18g^3 - 2gr^2 + 15g^2x + r^2x + 4gr(3g - 2x) \cos \phi) \sin \phi}{-9g^3 - 3gr^2 + 15g^2r \cos \phi + r^3 \cos \phi - 4gr^2 \cos^2 \phi + 4gr^2 \sin^2 \phi}. \end{aligned}$$

Thus,

$$x = \frac{2gr((9g^3 + 6gr^2) \cos \phi - r(9g^2 + r^2 + 6g^2 \cos 2\phi - gr \cos 3\phi))}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)}.$$

To find the y -coordinate of the symmedian point, we substitute this value of x into the equation of the symmedian from vertex A . Doing this, we see that

$$y = \frac{2g^2r(9g^2 + r^2 - 12gr \cos \phi + 2r^2 \cos 2\phi) \sin \phi}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)}.$$

Thus, the coordinates of the symmedian point are as follows:

$$\begin{aligned} K_\phi &= \left(\frac{2gr((9g^3 + 6gr^2) \cos \phi - r(9g^2 + r^2 + 6g^2 \cos 2\phi - gr \cos 3\phi))}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)}, \right. \\ &\quad \left. \frac{2g^2r(9g^2 + r^2 - 12gr \cos \phi + 2r^2 \cos 2\phi) \sin \phi}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)} \right). \end{aligned}$$

In order to prove that the symmedian point is always a fixed distance from the point K_n lying on ray \overrightarrow{OG} such that $OK_n = \frac{2gr^2}{r^2-g^2}$, we first recognize that \overrightarrow{OH} spans the Euler line, which, in our construction, is the x -axis. Since O is defined to be the origin, and H lies on the positive x -axis, K_n must have coordinates $(\frac{2gr^2}{r^2-g^2}, 0)$. We use the distance formula to find the distance between these two points.

$$\begin{aligned} & \left(\left(\frac{2gr((9g^3 + 6gr^2) \cos \phi - r(9g^2 + r^2 + 6g^2 \cos 2\phi - gr \cos 3\phi))}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)} - \frac{2gr^2}{r^2 - g^2} \right)^2 \right. \\ & \quad \left. + \left(\frac{2g^2r(9g^2 + r^2 - 12gr \cos \phi + 2r^2 \cos 2\phi) \sin \phi}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)} - 0 \right)^2 \right)^{\frac{1}{2}} \\ &= \sqrt{\frac{4g^4r^2}{(g^2 - r^2)^2}} \\ &= \sqrt{\frac{4g^4r^2}{(-1)^2(r^2 - g^2)^2}} \\ &= \frac{2g^2r}{r^2 - g^2}. \end{aligned}$$

Thus, the distance between K_n and K_ϕ is not dependent on ϕ , which implies that K_ϕ will always lie on the circle of radius $\frac{2g^2r}{r^2-g^2}$ centered at K_n . Additionally, it is nice to note that since r and g are distances, they are always strictly greater than zero. Moreover, since g cannot equal r , it follows that $\frac{2g^2r}{r^2-g^2} > 0$ and is always defined.

Let us now turn our attention to the second half of Theorem 3. We have three cases: $g < \frac{r}{3}$, $g = \frac{r}{3}$, and $g > \frac{r}{3}$.

Case 1. Let $g < \frac{r}{3}$. To show that every point on the Carleton Circle is the symmedian point of a triangle in Ω , we first reexamine the coordinates of the symmedian point. They are copied below for easy reference.

$$K_\phi = \left(\frac{2gr((9g^3 + 6gr^2) \cos \phi - r(9g^2 + r^2 + 6g^2 \cos 2\phi - gr \cos 3\phi))}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)}, \frac{2g^2r(9g^2 + r^2 - 12gr \cos \phi + 2r^2 \cos 2\phi) \sin \phi}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)} \right).$$

Let

$$A(\phi) = \frac{2gr((9g^3 + 6gr^2) \cos \phi - r(9g^2 + r^2 + 6g^2 \cos 2\phi - gr \cos 3\phi))}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)},$$

and let

$$B(\phi) = \frac{2g^2r(9g^2 + r^2 - 12gr \cos \phi + 2r^2 \cos 2\phi) \sin \phi}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)}.$$

Now, by definition, $K(\phi) = (A(\phi), B(\phi))$. We will show that $K(\phi)$ is continuous. Notice that $g^2 - r^2 \neq 0$ since $g \neq r$.² Moreover, $9g^2 + r^2 - 6gr \cos \phi \geq 9g^2 + r^2 - 6gr = (3g - r)^2 > 0$ since $g \neq \frac{r}{3}$. Thus, $(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi) \neq 0$. Since the numerators of both $A(\phi)$ and $B(\phi)$ are

²If g were equal to r , then this would force all three of the vertices of a $\triangle ABC$ to be coincident with G . However, a point is not a triangle.

combinations of continuous functions, it follows that both $A(\phi)$ and $B(\phi)$ are continuous, making $K(\phi) = (A(\phi), B(\phi))$ continuous.

Now, in Proposition 15, we saw that every triangle in a given gr -family (up to congruence) falls within certain bounds on ϕ . We will use these bounds for ϕ , so they are copied below for easy reference.

$$\cos^{-1}\left(\frac{3g+r}{2r}\right) \leq \phi \leq \cos^{-1}\left(\frac{3g-r}{2r}\right).$$

Setting $\phi = \cos^{-1}\left(\frac{3g+r}{2r}\right)$, we find using Mathematica that $K(\cos^{-1}\left(\frac{3g+r}{2r}\right)) = \left(\frac{2gr}{r-g}, 0\right)$. Similarly, setting $\phi = \cos^{-1}\left(\frac{3g-r}{2r}\right)$, we find that $K(\cos^{-1}\left(\frac{3g-r}{2r}\right)) = \left(\frac{2gr}{r+g}, 0\right)$.

Since the interval $[\cos^{-1}\left(\frac{3g+r}{2r}\right), \cos^{-1}\left(\frac{3g-r}{2r}\right)]$ is a connected set and $K(\phi)$ is continuous, it follows from the fact that continuous functions map connected sets to connected sets that the set $K([\cos^{-1}\left(\frac{3g+r}{2r}\right), \cos^{-1}\left(\frac{3g-r}{2r}\right)])$ is connected.³ Moreover, $K(\cos^{-1}\left(\frac{3g+r}{2r}\right)) = \left(\frac{2gr}{r-g}, 0\right)$ and $K(\cos^{-1}\left(\frac{3g-r}{2r}\right)) = \left(\frac{2gr}{r+g}, 0\right)$. These are the two points of intersection of the Carleton Circle with the Euler line. Since in the first part of the proof we showed that for all ϕ , $K(\phi)$ lies on the Carleton Circle, it follows that the only way $K([\cos^{-1}\left(\frac{3g+r}{2r}\right), \cos^{-1}\left(\frac{3g-r}{2r}\right)])$ can be a connected set is if it encompasses minimally either the upper or lower half of the Carleton Circle (including the intersections of the Carleton Circle with the Euler line). In other words, $K([\cos^{-1}\left(\frac{3g+r}{2r}\right), \cos^{-1}\left(\frac{3g-r}{2r}\right)])$ must be at least $\{(x, y) \mid (x - \frac{2gr^2}{r^2-g^2})^2 + y^2 = \frac{4g^4r^2}{(r^2-g^2)^2} \wedge \text{either } y \geq 0 \text{ or } y \leq 0\}$. To show that $K([\cos^{-1}\left(\frac{3g+r}{2r}\right), \cos^{-1}\left(\frac{3g-r}{2r}\right)])$ encompasses *only* one half of the Carleton Circle, we use Mathematica to take the derivative of $A(\phi)$.

$$A'(\phi) = \frac{6g^2r(27g^4 + 36g^2r^2 + r^4 - 4gr(18g^2 + 5r^2)\cos\phi + 2r^2(15g^2 + r^2)\cos 2\phi - 4gr^3\cos 3\phi)\sin\phi}{(r^2 - g^2)(9g^2 + r^2 - 6gr\cos\phi)^2}.$$

Setting $A'(\phi)$ equal to 0, we find the following critical points of $A(\phi)$ in $[0, 2\pi]$:

$$\begin{aligned} \phi &= 0 \\ \phi &= \pi \\ \phi &= \pm \cos^{-1}\left(\frac{3g+r}{2r}\right) \\ \phi &= \pm \cos^{-1}\left(\frac{3g-r}{2r}\right) \\ \phi &= \pm \cos^{-1}\left(\frac{3g^2+r^2}{4gr}\right) \end{aligned}$$

Certainly $\pm \cos^{-1}\left(\frac{3g\pm r}{2r}\right) \notin (\cos^{-1}\left(\frac{3g+r}{2r}\right), \cos^{-1}\left(\frac{3g-r}{2r}\right))$. Moreover, for $\pm \cos^{-1}\left(\frac{3g^2+r^2}{4gr}\right)$ to exist, $-1 \leq \frac{3g^2+r^2}{4gr} \leq 1$. Thus,

³See Munkres, pg 150.

$$\begin{aligned}
\frac{3g^2 + r^2}{4gr} &\leq 1 \\
3g^2 + r^2 &\leq 4gr \\
3g^2 + r^2 - 4gr &\leq 0 \\
(3g - r)(g - r) &\leq 0
\end{aligned}$$

This implies that either

1. $3g - r \leq 0 \wedge g - r \geq 0$ or
2. $3g - r \geq 0 \wedge g - r \leq 0$.

However, $g - r \geq 0 \rightarrow g \geq r$, which cannot occur. Moreover, $3g - r \geq 0 \rightarrow 3g \geq r$, which contradicts our assumption that $g < \frac{r}{3}$. Thus, $\pm \cos^{-1}(\frac{3g^2+r^2}{4gr})$ is not a valid critical number for $g < \frac{r}{3}$.

Finally, recall that $\cos^{-1}(\frac{3g+r}{2r})$ and $\cos^{-1}(\frac{3g-r}{2r})$ were found by constructing the two isosceles triangles in a gr -family and measuring the positive $\angle GOP$ where P is one of the two vertices of $\triangle ABC$ not on the Euler line of the gr -family. Since one vertex of $\triangle ABC$ on the Euler line always forces the remaining two vertices to lie on opposite sides of the Euler line, it follows that $m\angle GOP \in (0, \pi)$. Thus, $[\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})] \subset (0, \pi)$. This implies that $0, \pi \notin [\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})]$.

Therefore, $A(\phi)$ is monotone on $[\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})]$. This in turn indicates that $K([\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})])$ can only be one half of the Carleton Circle.

To determine whether that half is the upper or lower half, we find the derivative of $B(\phi)$.

$$B'(\phi) = \frac{6g^2r(3g^2(9g^2+7r^2)\cos\phi+r(-4g(9g^2+2r^2)\cos 2\phi+r(15g^2+r^2)\cos 3\phi-2g(9g^2+r^2)\cos 4\phi)}{(g^2-r^2)(9g^2+r^2-6gr\cos\phi)^2}.$$

Evaluating B' at $\phi = \cos^{-1}(\frac{3g+r}{2r})$, we find that $B'(\cos^{-1}(\frac{3g+r}{2r})) = \frac{6g^2}{r-g} > 0$. Thus, at $\phi = \cos^{-1}(\frac{3g+r}{2r})$, $B(\phi)$ is increasing, which implies that $K([\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})])$ is the upper half of the Carleton Circle. In other words, $K([\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})]) = \{(x, y) \mid (x - \frac{2gr^2}{r^2-g^2})^2 + y^2 = \frac{4g^4r^2}{(r^2-g^2)^2} \wedge \text{either } y \geq 0\} = C_1$.

Thus, if $(x, y) \in C_1$, it follows that there exists a $\triangle ABC_\phi$ in Ω , $\phi \in [\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})]$, with symmedian point (x, y) . If (x, y) is a point on the lower half of the Carleton Circle, then by symmetry it will be the symmedian point of $\triangle ABC_{-\phi}$ where $(x, -y)$ is the symmedian point of $\triangle ABC_\phi$, $\phi \in [\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})]$.

Therefore, every point on the Carleton Circle is the symmedian point of a triangle in Ω .

Case 2. Let $g = \frac{r}{3}$. This case was covered in Lemma 2.

Case 3. Let $g > \frac{r}{3}$. To show that for all $(x, y) \in \{(x, y) \mid (x - \frac{2gr^2}{r^2-g^2})^2 + y^2 = \frac{4g^4r^2}{(r^2-g^2)^2} \wedge x^2 + y^2 < r^2\}$, there exists a triangle in Ω with symmedian point (x, y) , we follow a method similar to that of Case 1.

Now, for the case in which $g > \frac{r}{3}$, Proposition 17 states that every triangle in a given gr -family (up to congruence) falls within the following bounds on ϕ :

$$\cos^{-1}\left(\frac{3g-r}{2r}\right) \leq \phi < \cos^{-1}\left(\frac{3g^2-r^2}{2rg}\right).$$

As in Case 1, we will use these bounds. We saw earlier that $K(\cos^{-1}(\frac{3g-r}{2r})) = (\frac{2gr}{r+g}, 0)$. Using Mathematica, we find that

$$K(\cos^{-1}(\frac{3g^2-r^2}{2rg})) = \left(\frac{3g^2+r^2}{4g}, -\frac{1}{4}r\sqrt{10 - \frac{9g^2}{r^2} - \frac{r^2}{g^2}} \right).$$

Since the y -coordinate of $K(\cos^{-1}(\frac{3g^2-r^2}{2gr}))$ involves a square root, we must verify that $\cos^{-1}(\frac{3g^2-r^2}{2gr})$ is actually in the domain of $K(\phi)$. To do this, we note that

$$\begin{aligned} -\frac{1}{4}r\sqrt{10 - \frac{9g^2}{r^2} - \frac{r^2}{g^2}} &= -\frac{1}{4}r\sqrt{10 - \frac{9g^4+r^4}{g^2r^2}} \\ &= -\frac{1}{4}r\sqrt{10 - \frac{9g^4+r^4+6r^2g^2-6r^2g^2}{r^2g^2}} \\ &= -\frac{1}{4}r\sqrt{10 - \frac{(3g^2+r^2)^2-6r^2g^2}{r^2g^2}} \\ &= -\frac{1}{4}r\sqrt{10 - \left(\frac{(3g^2+r^2)^2}{r^2g^2} - \frac{6r^2g^2}{r^2g^2}\right)} \\ &= -\frac{1}{4}r\sqrt{16 - \frac{(3g^2+r^2)^2}{r^2g^2}}. \end{aligned}$$

We need to show that $\frac{(3g^2+r^2)^2}{r^2g^2} \leq 16$. Thus, we examine the conditions under which this inequality holds.

$$\begin{aligned} \frac{(3g^2+r^2)^2}{r^2g^2} &\leq 16 \\ (3g^2+r^2)^2 &\leq 16r^2g^2 \\ 3g^2+r^2 &\leq 4rg \\ 3g^2-4gr+r^2 &\leq 0 \\ (3g-r)(g-r) &\leq 0 \end{aligned}$$

This implies that either

1. $3g-r \leq 0 \wedge g-r \geq 0$ or
2. $3g-r \geq 0 \wedge g-r \leq 0$.

In the first case, $g-r \geq 0 \rightarrow g \geq r$, which is a contradiction. In the second, $3g-r \geq 0 \rightarrow g \geq \frac{r}{3}$, which holds by assumption, and $g-r \leq 0 \rightarrow g \leq r$, which is true. Thus, we see that $-\frac{1}{4}r\sqrt{10 - \frac{9g^2}{r^2} - \frac{r^2}{g^2}}$ will be a real number for all $g \geq \frac{r}{3}$.

Now, again by the preservation of connectedness under continuous functions, since $[\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]$ is connected, $K([\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})])$ is connected. This implies that $K([\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})])$ minimally contains $\{(x, y) \mid (x - \frac{2gr^2}{r^2-g^2})^2 + y^2 = \frac{4g^4r^2}{(r^2-g^2)^2} \wedge y \leq 0\}$. However, from Case 1, we know that $A(\phi)$ has the following critical points in $[0, 2\pi]$:

$$\begin{aligned}\phi &= 0 \\ \phi &= \pi \\ \phi &= \pm \cos^{-1}\left(\frac{3g+r}{2r}\right) \\ \phi &= \pm \cos^{-1}\left(\frac{3g-r}{2r}\right) \\ \phi &= \pm \cos^{-1}\left(\frac{3g^2+r^2}{4gr}\right)\end{aligned}$$

Since $\frac{3g+r}{2r} > \frac{r+r}{2r} = 1$, we see that $\pm \cos^{-1}(\frac{3g+r}{2r})$ are not real numbers, and thus are not valid critical points. By construction $[\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})] \subseteq (0, \pi)$. This implies that $0, \pi \notin [\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]$. Moreover, by the following argument, $\cos^{-1}(\frac{3g^2+r^2}{4gr}) \in (0, \pi)$:

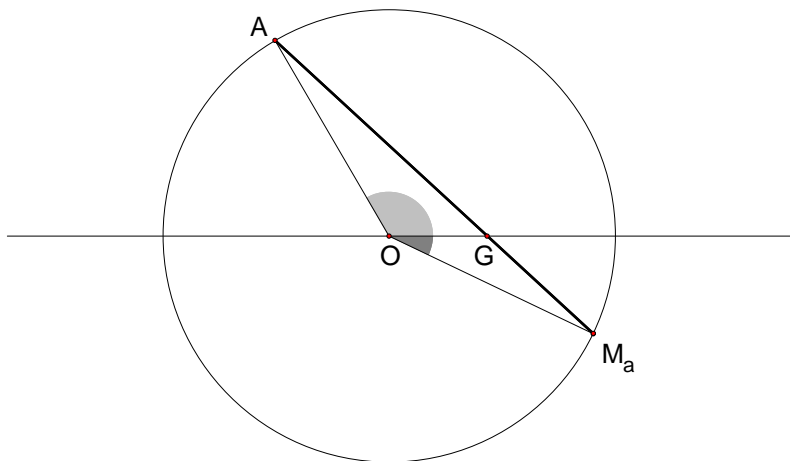


Figure 29: The angles $\cos^{-1}(\frac{3g^2-r^2}{2rg})$ and $\cos^{-1}(\frac{3g^2+r^2}{4gr})$.

This proof utilizes Figure 29 above. Let Ω^* be a gr -family of triangles in which $g > \frac{r}{3}$. Then, it follows that there exists an $\angle GOA$ such that the midpoint M_a of \overline{BC} lies on the circumcircle of Ω^* . As discussed in Proposition 17, the measure of this angle is $\cos^{-1}(\frac{3g^2-r^2}{2gr})$. We will show that $m\angle GOM_a = \cos^{-1}(\frac{3g^2+r^2}{4gr})$.

To begin, construct the segment $\overline{AM_a}$ where $m\angle GOA = \cos^{-1}(\frac{3g^2-r^2}{2gr})$. Then, by the law of cosines,

$$\begin{aligned}
AG^2 &= r^2 + g^2 - 2gr \cos(\cos^{-1}(\frac{3g^2 - r^2}{2gr})) \\
&= r^2 + g^2 - 2gr \frac{3g^2 - r^2}{2gr} \\
&= r^2 + g^2 - 3g^2 + r^2 \\
&= 2r^2 - 2g^2 .
\end{aligned}$$

So, $AG = \sqrt{2(r^2 - g^2)}$. Now, by Proposition 4, $AG = 2GM_a$. Thus, $GM_a = \frac{\sqrt{2(r^2 - g^2)}}{2}$. To find $m\angle GOM_a$, we again use the law of cosines.

$$\begin{aligned}
GM^2 &= g^2 + r^2 - 2gr \cos(m\angle GOM_a) \\
\left(\frac{\sqrt{2(r^2 - g^2)}}{2}\right)^2 &= g^2 + r^2 - 2gr \cos(m\angle GOM_a) \\
\frac{r^2 - g^2}{2} &= g^2 + r^2 - 2gr \cos(m\angle GOM_a) \\
r^2 - g^2 &= 2g^2 + 2r^2 - 4gr \cos(m\angle GOM_a) \\
-r^2 - 3g^2 &= -4gr \cos(m\angle GOM_a) \\
\frac{3g^2 + r^2}{4gr} &= \cos(m\angle GOM_a) \\
\cos^{-1}\left(\frac{3g^2 + r^2}{4gr}\right) &= m\angle GOM_a .
\end{aligned}$$

Now, notice that the only instance in which $m\angle GOA$ can equal π occurs when $g = \frac{r}{3}$. Thus, since $g > \frac{r}{3}$, it follows that $m\angle GOA < \pi$. However, $m\angle GOA < \pi$ indicates that M_a and A are on opposite sides of the Euler line of Ω^* . Consequently, $\angle GOM_a \in (0, \pi)$.

Now that we have established that both $\cos^{-1}(\frac{3g^2 + r^2}{4gr})$ and $\cos^{-1}(\frac{3g - r}{2r})$ lie in the interval $(0, \pi)$, we can state the following:

$$\begin{aligned}
(3g + r)(g - r) &< 0 \\
3g^2 - 2gr - r^2 &< 0 \\
3g^2 - 2gr &< r^2 \\
6g^2 - 2gr &< 3g^2 + r^2 \\
2g(3g - r) &< 3g^2 + r^2 \\
3g - r &< \frac{3g^2 + r^2}{2g} \\
\frac{3g - r}{2r} &< \frac{3g^2 + r^2}{4gr} \\
\cos^{-1}\left(\frac{3g - r}{2r}\right) &> \cos^{-1}\left(\frac{3g^2 + r^2}{4gr}\right) .
\end{aligned}$$

Since $\cos^{-1}(\frac{3g-r}{2r}) \leq \cos^{-1}(\frac{3g^2-r^2}{2rg})$, we have now established that there are no critical points of $A(\phi)$ on $(\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg}))$. Thus, $A(\phi)$ is monotone on $[\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]$. Since $B(\cos^{-1}(\frac{3g^2-r^2}{2rg})) = -\frac{1}{4}r\sqrt{16 - \frac{(3g^2+r^2)^2}{r^2g^2}}$, this implies that $K([\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})])$ is an arc of the Carleton Circle lying below the Euler line. Thus, $K([\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]) = \{(x, y) \mid (x - \frac{2gr^2}{r^2-g^2})^2 + y^2 = \frac{4g^4r^2}{(r^2-g^2)^2} \wedge y \leq 0\}$.

Now, as discussed in a footnote in the proof of Lemma 2, the symmedian point of a given $\triangle ABC$ can never fall on the circumcircle of $\triangle ABC$. Using Mathematica, we find the intersections of the circumcircle of Ω with the Carleton Circle to be the points $(\frac{3g^2+r^2}{4g}, \frac{1}{4}r\sqrt{10 - \frac{9g^2}{r^2} - \frac{r^2}{g^2}})$ and $(\frac{3g^2+r^2}{4g}, -\frac{1}{4}r\sqrt{10 - \frac{9g^2}{r^2} - \frac{r^2}{g^2}})$. Thus, $K(\cos^{-1}(\frac{3g^2-r^2}{2rg})) = (\frac{3g^2+r^2}{4g}, -\frac{1}{4}r\sqrt{10 - \frac{9g^2}{r^2} - \frac{r^2}{g^2}})$ is not actually a valid symmedian point even though it is in the domain of K . With this in mind, we restrict our original interval to $[\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]$. Since $[\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})] \subseteq [\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]$, it follows that $K([\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]) \subseteq K([\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})])$. Therefore, $K([\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})])$ will be the connected set $\{(x, y) \mid (x - \frac{2gr^2}{r^2-g^2})^2 + y^2 = \frac{4g^4r^2}{(r^2-g^2)^2} \wedge \frac{2gr}{g+r} \leq x < \frac{3g^2+r^2}{4g} \wedge y \leq 0\} = C_2$.

Thus, if $(x, y) \in C_2$, it follows that there exists a $\triangle ABC_\phi$ in Ω , $\phi \in [\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]$, with symmedian point (x, y) . If (x, y) is a point on the reflection of C_2 over the Euler line of Ω , then by symmetry it will be the symmedian point of $\triangle ABC_{-\phi}$ where $(x, -y)$ is the symmedian point of $\triangle ABC_\phi$, $\phi \in [\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]$.

Therefore, every point on the arc of the Carleton Circle $\{(x, y) \mid (x - \frac{2gr^2}{r^2-g^2})^2 + y^2 = \frac{4g^4r^2}{(r^2-g^2)^2} \wedge \frac{2gr}{g+r} \leq x < \frac{3g^2+r^2}{4g}\}$ is the symmedian point of a triangle in Ω .

Finally, to ensure that we have not missed any points on the Carleton Circle which fall within the circumcircle, we recall from above that the intersections of the two circles occur at $x = \frac{3g^2+r^2}{4g}$. This implies that for any $x > \frac{3g^2+r^2}{4g}$, if (x, y) is a point on the Carleton Circle, then (x, y) will lie outside the circumcircle of Ω . Thus, we have discussed all points on the Carleton Circle which could be symmedian points. For all points (x, y) on the Carleton Circle and inside the circumcircle of Ω , there exists a $\triangle ABC \in \Omega$ with symmedian point (x, y) . \square

It is indeed interesting that the symmedian points of a gr -family of triangles form a complete circle or a segment of a circle. To extend on this idea, we have the next corollary.

Corollary 3. *Every point on \overline{GK} traces a circle as ϕ varies.*

Proof. Let triangle $\triangle ABC$ be given with centroid G , circumcenter O , symmedian point K , and Knights' Point K_n . Consider the vector \overrightarrow{GK} . Vector \overrightarrow{GK} spans the line \overline{GK} . Thus, for all points P on \overline{GK} , $\overrightarrow{GP} = a\overrightarrow{GK}$, where $a \in \mathbb{R}$. Let K_1 and K_2 be the intersections of the Carleton Circle with the Euler line of $\triangle ABC$. Then, let P_1 and P_2 be two points on the Euler line such that $\overrightarrow{GP_1} = a\overrightarrow{GK_1}$ and $\overrightarrow{GP_2} = a\overrightarrow{GK_2}$. Now, since $a = \frac{GP}{GK} = \frac{GP_1}{GK_1}$, and angle $\angle KGK_1$ is common, it follows that $\triangle KGK_1 \sim \triangle PGP_1$. By the same argument, $\triangle KGK_2 \sim \triangle PGP_2$. Thus, $\frac{PP_1}{KK_1} = \frac{GP}{GK} = \frac{PP_2}{KK_2}$. Moreover, $P_1P_2 = GP_2 - GP_1 = aGK_2 - aGK_1 = a(GK_2 - GK_1) = aK_1K_2$. This implies that $\frac{P_1P_2}{K_1K_2} = a$. Thus, by SSS similarity, $\triangle PP_1P_2 \sim \triangle KK_1K_2$. Since K lies on the circle with diameter $\overline{K_1K_2}$, $m\angle K_1KK_2 = \frac{\pi}{2}$. Thus, $m\angle P_1PP_2 = \frac{\pi}{2}$. This implies that P lies on the circle with diameter $\overline{P_1P_2}$. We will call this circle the Carleton a -Circle.

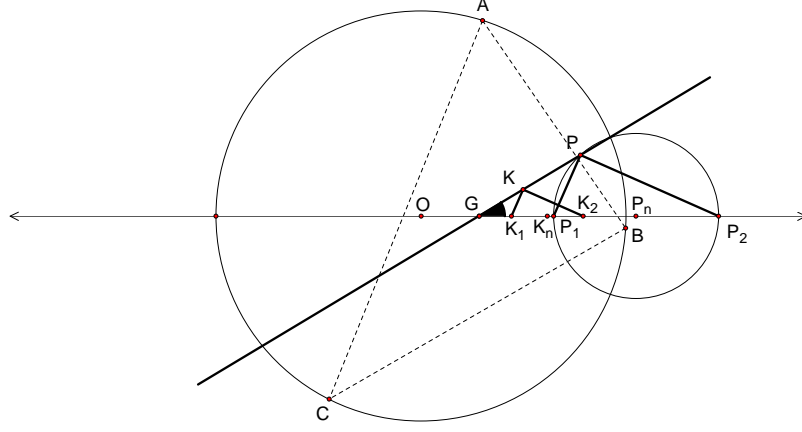


Figure 30: Every point on \overline{GK} lies on a circle centered on the Euler line of a given gr -family of triangles.

Now, the center P_n of the Carleton a Circle is the point located halfway between P_1 and P_2 . So, $GP_n = GP_1 + P_1P_n = aGK_1 + \frac{P_1P_2}{2} = aGK_1 + \frac{aK_1K_2}{2} = aGK_1 + aK_1K_n = aGK_n$. If we consider the Euler line of triangle $\triangle ABC$ to be the x -axis and O to be the origin, then $OP_n = g + GP_n = g + aGK_n = g + a(\frac{2gr^2}{r^2-g^2} - g) = g + a\frac{2gr^2}{r^2-g^2} - ag = (1-a)g + a\frac{2gr^2}{r^2-g^2}$.

Now, by the proof of Theorem 3, we know that every point on the Carleton Circle which lies inside the circumcircle of $\triangle ABC$ corresponds to a symmedian point of some triangle in the gr -family of $\triangle ABC$. Define the function $h : \overrightarrow{GK}_\phi \Rightarrow \overrightarrow{GP}_\phi$ such that $h(\overrightarrow{GK}_\phi) = a\overrightarrow{GK}_\phi = \overrightarrow{GP}_\phi$. Now, $K(\phi)$ is a continuous function, or, simply put, the coordinates of K_ϕ are continuous. Moreover, the components of \overrightarrow{GK}_ϕ are found through subtraction of g and 0 from

$$\frac{2gr((9g^3 + 6gr^2) \cos \phi - r(9g^2 + r^2 + 6g^2 \cos 2\phi - gr \cos 3\phi))}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)}$$

and

$$\frac{2g^2r(9g^2 + r^2 - 12gr \cos \phi + 2r^2 \cos 2\phi) \sin \phi}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \phi)},$$

respectively. Thus, the components of \overrightarrow{GK}_ϕ are continuous. Since the function h just multiplies the vector \overrightarrow{GK}_ϕ by a real number, it follows that h must be continuous. Thus, since continuous functions map connected sets to connected sets, by an argument similar to that in the proof of Theorem 3, if $g < \frac{r}{3}$, then P will trace the entire Carleton a -Circle. If $g = \frac{r}{3}$, P will trace the entire Carleton a -Circle except for the farthest intersection of the Carleton a -Circle with the Euler line of the gr -family. If $g > \frac{r}{3}$, then P will trace an arc whose endpoints correspond to the places at which the Carleton Circle meets the circumcircle of the gr -family.

Finally, the center P_n of the Carleton a -Circle is the point located halfway between P_1 and P_2 . So, $GP_n = GP_1 + P_1P_n = aGK_1 + \frac{P_1P_2}{2} = aGK_1 + \frac{aK_1K_2}{2} = aGK_1 + aK_1K_n = aGK_n$. If we consider the Euler line of triangle $\triangle ABC$ to be the x -axis and O to be the origin, then $OP_n = g + GP_n = g + aGK_n = g + a(\frac{2gr^2}{r^2-g^2} - g) = g + a\frac{2gr^2}{r^2-g^2} - ag = (1-a)g + a\frac{2gr^2}{r^2-g^2}$. □

7 The Sum of Squares

Now that we have examined what happens to certain points when we vary $\angle GOA$ in a gr -family, let us turn to another interesting property that naturally emerges from our construction.

Proposition 31. *For any point P on $\triangle ABC$'s plane, define function $\mathcal{D} = PA^2 + PB^2 + PC^2$. Then \mathcal{D} has constant values for the gr -family of triangles, and the locus of points sharing constant \mathcal{D} -values forms circles centered at the centroid G .*

Proof. To show this is actually true, we build upon the coordinate system we placed $\triangle ABC$ in. Following our previous definitions, point A 's polar coordinates are $(r \cos \phi, r \sin \phi)$. Set point P 's polar coordinates to be $(x, y) = (s \cos \beta, s \sin \beta)$. Next we just need to prove \mathcal{D} is independent of ϕ . Recall in the previous section we have found the coordinates for the three vertices of $\triangle ABC$. Here, we place them into function \mathcal{D} . After simplification, we get the following results:

$$\mathcal{D} = 3(r^2 + s^2 - 2gs \cos \beta) .$$

Notice in this case the function \mathcal{D} is indeed independent of ϕ . If we convert it into Cartesian coordinates and treat \mathcal{D} as a constant D , we get an equation of circles centered at $G(g, 0)$ for any $D > 3r^2 + 3g^2$:

$$(x - g)^2 + y^2 = \frac{D}{3 - r^2 - g^2} ,$$

which completes our proof. □

8 Conclusion

We have achieved our objective of finding a natural way to relate triangles. We created a construction based on the Euler line, an element intrinsic to a triangle. Using that construction, we parameterized triangles and grouped them into families. Then, we bounded the construction, enabling us to visualize triangle space. In order to algebraically conceive of distances between triangles and to account for certain peculiarities in our space, we created a metric. Then, we looked at the loci of certain triangle centers and special points in our triangle families, discovering for example that a gr -family has a constant Nine-Point Circle. We also looked at the symmedian point in a gr -family and proved that it will always lie on a circle (the Carleton Circle) centered at the Knights' Point on the Euler line of that family. Finally, we noticed that given a point in the plane, the sum of squares of distances from that point to the vertices of any triangle in a gr -family is constant.

While the work here is a basis for this way of conceiving triangles, it is certainly not exhaustive. Euclidean Geometry has been a branch of mathematics for thousands of years. Despite this, there is clearly still much to discover.

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9 Appendix A

In this appendix, we find explicit representations for the lengths of the sides of the triangle in our construction.

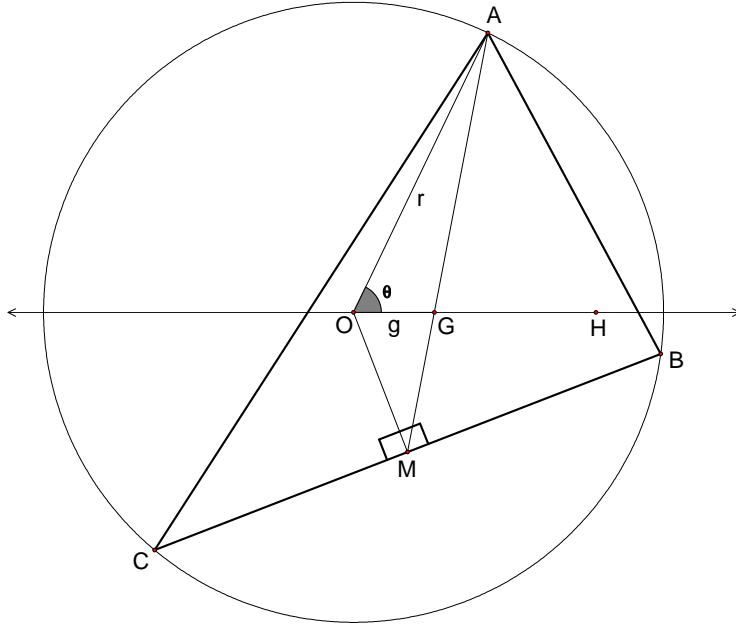


Figure 31: Finding Side Lengths.

First, we will find the length of \overline{BC} . Using Law of Cosines on $\triangle AOG$:

$$AG^2 = r^2 + g^2 - 2rg \cos \theta \Rightarrow AM = \frac{3}{2} \sqrt{r^2 + g^2 - 2rg \cos \theta}$$

Using the Law of Sines on $\triangle AOG$:

$$\begin{aligned} \frac{\sin \angle OAG}{g} &= \frac{\sin \theta}{\sqrt{r^2 + g^2 - 2rg \cos \theta}} \Rightarrow \sin \angle OAG = \frac{g \sin \theta}{\sqrt{r^2 + g^2 - 2rg \cos \theta}} \\ \Rightarrow \cos \angle OAG &= \frac{r - g \cos \theta}{\sqrt{r^2 + g^2 - 2rg \cos \theta}} \end{aligned}$$

Now:

$$\begin{aligned} OM^2 &= AM^2 + AO^2 - 2(AM)(AO) \cos \angle OAG \\ \Rightarrow OM^2 &= \frac{9}{4}(r^2 + g^2 - 2rg \cos \theta) + r^2 - 3r \sqrt{r^2 + g^2 - 2rg \cos \theta} \cdot \frac{r - g \cos \theta}{r^2 + g^2 - 2rg \cos \theta} \\ &\Rightarrow OM^2 = \frac{1}{4}(r^2 + 9g^2 - 6rg \cos \theta) \\ &\Rightarrow OM = \frac{1}{2} \sqrt{r^2 + 9g^2 - 6rg \cos \theta} \\ \Rightarrow BM^2 &= r^2 - \left(\frac{1}{4}r^2 - 9g^2 + 6rg \cos \theta \right) \\ &\Rightarrow BM^2 = \frac{3}{4}(r^2 - 3g^2 + 2rg \cos \theta) \end{aligned}$$

$$\Rightarrow BM = \frac{1}{2}\sqrt{3(r^2 - 3g^2 + 2rg \cos \theta)}$$

$$BC = 2BM$$

$$\therefore BC = \sqrt{3(r^2 - 3g^2 + 2rg \cos \theta)}$$

Next, we will find the length of \overline{AB} :

$$OG^2 = OM^2 + MG^2 - 2(OM)(MG) \cos \angle OMG$$

$$g^2 = \frac{r^2}{4} + \frac{9g^2}{4} - \frac{6rg \cos \theta}{4} + \frac{r^2}{4} + \frac{g^2}{4} - \frac{2rg \cos \theta}{4} - \frac{1}{2}\sqrt{r^2 + g^2 - 2rg \cos \theta}\sqrt{r^2 + 9g^2 - 6rg \cos \theta} \cdot \cos \angle OMG$$

$$\Rightarrow \cos \angle OMG = \frac{r^2 + 3g^2 - 4rg \cos \theta}{\sqrt{r^2 + g^2 - 2rg \cos \theta}(r^2 + 9g^2 - 6rg \cos \theta)}$$

$$\Rightarrow \angle OMG = \cos^{-1} \left(\frac{r^2 + 3g^2 - 4rg \cos \theta}{\sqrt{r^2 + g^2 - 2rg \cos \theta}(r^2 + 9g^2 - 6rg \cos \theta)} \right)$$

$$\Rightarrow \angle AMB = \frac{\pi}{2} - \cos^{-1} \left(\frac{r^2 + 3g^2 - 4rg \cos \theta}{\sqrt{r^2 + g^2 - 2rg \cos \theta}(r^2 + 9g^2 - 6rg \cos \theta)} \right)$$

$$\Rightarrow \cos \angle AMB = \sin \left[\cos^{-1} \left(\frac{r^2 + 3g^2 - 4rg \cos \theta}{\sqrt{r^2 + g^2 - 2rg \cos \theta}(r^2 + 9g^2 - 6rg \cos \theta)} \right) \right]$$

Using a right triangle:

$$x_2^2 = (r^2 + g^2 - 2rg \cos \theta)(r^2 + 9g^2 - 6rg \cos \theta) - (r^2 + 3g^2 - 4rg \cos \theta)^2$$

$$(x_2)^2 = 4r^2g^2 \sin^2 \theta$$

$$\Rightarrow x_2 = 2rg \sin \theta$$

$$\Rightarrow \cos \angle AMB = \frac{2rg \sin \theta}{\sqrt{(r^2 + g^2 - 2rg \cos \theta)(r^2 + 9g^2 - 6rg \cos \theta)}}$$

$$\Rightarrow AB^2 = AM^2 + BM^2 - 2(AM)(BM) \cos \angle AMB$$

$$\Rightarrow AB^2 = \frac{9}{4}r^2 - \frac{9}{2}rg \cos \theta + \frac{3}{4}r^2 + \frac{3}{2}rg \cos \theta - 3\sqrt{3(r^2 - 3g^2 + 2rg \cos \theta)} \cdot \left(\frac{rg \sin \theta}{r^2 + 9g^2 - 6rg \cos \theta} \right)$$

$$\Rightarrow AB^2 = 3r^2 - 3rg \cos \theta - 3rg \sin \theta \cdot \frac{\sqrt{3r^2 - 9g^2 + 6rg \cos \theta}}{\sqrt{r^2 + 9g^2 - 6rg \cos \theta}}$$

$$\therefore AB = \sqrt{3r^2 - 3rg \cos \theta - 3rg \sin \theta \cdot \frac{\sqrt{3r^2 - 9g^2 + 6rg \cos \theta}}{\sqrt{r^2 + 9g^2 - 6rg \cos \theta}}}$$

Finally, we will find the length of \overline{AC} :

$$\angle AMC = \frac{\pi}{2} + \angle OMG$$

$$\cos \angle AMC = \sin \left[\cos^{-1} \frac{r^2 + 3g^2 - 4rg \cos \theta}{\sqrt{(r^2 + g^2 - 2rg \cos \theta)(r^2 + 9g^2 - 6rg \cos \theta)}} \right]$$

$$\Rightarrow \cos \angle AMC = \frac{2rg \sin \theta}{\sqrt{(r^2 + g^2 - 2rg \cos \theta)(r^2 + 9g^2 - 6rg \cos \theta)}}$$

$$\therefore AC = \sqrt{3r^2 - 3rg \cos \theta + 3rg \sin \theta \cdot \frac{\sqrt{3r^2 - 9g^2 + 6rg \cos \theta}}{\sqrt{r^2 + 9g^2 - 6rg \cos \theta}}}$$