The Pfaffian Transformation

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Abstract

We define a function on sequences, which we call the Pfaffian transformation, based on the concept of the Pfaffian of a skew-symmetric matrix. We discuss properties of the Pfaffian transformation, culminating in a theorem proving that the Pfaffian transformation maps sequences satisfying linear recurrences to sequences satisfying linear recurrences.

1 Introduction to the Pfaffian Transformation

Given a sequence $(a_0, a_1, a_2, ...)$, we define a sequence of skew-symmetric matrices $(A_0, A_1, A_2, ...)$ where A_n is
$$\begin{pmatrix} 0 & a_0 & a_1 & a_2 & \cdots & a_{2n} \\ -a_0 & 0 & a_0 & a_1 & \cdots & a_{2n-1} \\ -a_1 & -a_0 & 0 & a_0 & \cdots & a_{2n-2} \\ -a_2 & -a_1 & -a_0 & 0 & \cdots & a_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{2n} & -a_{2n-1} & -a_{2n-2} & -a_{2n-3} & \cdots & 0 \end{pmatrix}.$$

We then define the *Pfaffian transformation* of $(a_0, a_1, a_2, ...)$ to be the sequence of Pfaffians $(Pf(A_0), Pf(A_1), Pf(A_2), ...)$, where the *Pfaffian* of a skew-symmetric matrix is the positive or negative square root of its determinant. (For a precise definition of the Pfaffian, see section 2.) The Pfaffian transformation is thus a function from sequences to sequences.

We begin by observing the effects of this transformation on some simple, well-known sequences:

$$\begin{aligned} (1, 2, 3, 4, 5, ...) &\rightarrow (1, 0, 0, 0, 0, ...) \\ (2, 2, 2, 2, 2, ...) &\rightarrow (2, 4, 8, 16, 32, ...) \\ (1, 2, 4, 8, 16, ...) &\rightarrow (1, 1, 1, 1, 1, ...) \\ (1, 1, 2, 3, 5, ...) &\rightarrow (1, 2, 4, 8, 16, ...) \end{aligned}$$

Observe that each of these input sequences satisfies a linear recurrence relation and is mapped to a sequence that also satisfies a linear recurrence. Our main result proves that this holds in general; we also give an algorithm to calculate the output recurrence for such an input sequence.

In section 2 we provide a definition of the Pfaffian of a matrix and prove some useful properties. In section 3 we then use these to justify the above examples and develop a useful reduction technique that greatly simplifies the process for evaluating the Pfaffian transformation of a sequence that satisfies a linear recurrence. In section 4 we extend this technique and prove that it works in general. Our main result is located in section 7, relying on the technique of section 4 and the definitions of section 6; section 5 provides motivation for the result via proofs of some simple cases. Finally, in sections 8 and 9 we provide useful techniques for evaluating finite portions of the Pfaffian transformation of a sequence via computer and in section 10 we note some conjectures raised by our studies of the Pfaffian transformation.

2 The Pfaffian of a Skew-Symmetric Matrix

We will first define a *skew-symmetric matrix*, A, as a matrix in which $A^T = -A$. We will only concern ourselves with the Pfaffian of $2n \times 2n$ skew-symmetric matrices as it turns out that the Pfaffian of a $(2n+1) \times (2n+1)$ skew-symmetric matrix is always zero.

The determinant of a skew-symmetric matrix, A, can be written as the square of a polynomial in the matrix entries. This polynomial is called the Pfaffian of A. For example, if

$$A = \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{pmatrix}$$

then

$$Pf(A) = b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23}.$$

To define the Pfaffian more formally, we first consider a generic $2n \times 2n$ skew-symmetric matrix with zeros on the diagonal, as this is the type of matrix that we will construct in evaluating the Pfaffian transformation.

$$A = \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,2n} \\ -a_{1,2} & 0 & \cdots & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{2n,1} & -a_{2n,2} & \cdots & 0 \end{pmatrix}$$

We now define the Pfaffian.

Definition 2.1. *The Pfaffian of a $2n \times 2n$ skew-symmetric matrix A is defined as*

$$\begin{aligned} Pf(A) &= \sum_{\pi} wt(\pi) \\ &= \sum_{\pi} sgn(\pi) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n} \\ &= \sum_{\pi} sgn(\pi) a_{\pi(1)\pi(2)} a_{\pi(3)\pi(4)} \cdots a_{\pi(2n-1)\pi(2n)} \\ &= \sum_{\pi} sgn(\pi) \prod_{i=1}^{n} a_{\pi(2i-1)\pi(2i)}, \end{aligned}$$

where the sums are over the set of all permutations π on the set $\{1, 2, ..., 2n\}$ of the form

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 2n \\ i_1 & j_1 & i_2 & j_2 & \cdots & j_n \end{bmatrix}$$

for $i_k < j_k$ and $i_1 < i_2 < \cdots < i_n$. The sign of a permutation is given by $sgn(\pi) = (-1)^{inv(\pi)}$.

Recall the Leibniz formula for the determinant.

$$Det(A) = \sum_{\pi} sgn(\pi) \prod a_{n\pi(n)} \tag{1}$$

Note that the definition of the Pfaffian is similar to that of the determinant. In fact, as previously noted, the determinant and the Pfaffian have a specific relation. This is a classic result that was proved by Arthur Cayley (1).

Proposition 2.2. $Pf(A)^2 = \det(A)$, *where A is a $2n \times 2n$ skew symmetric matrix.*

A second important identity involving the Pfaffian is given in the following proposition.

Proposition 2.3. $Pf(BAB^T) = \det(B) Pf(A)$ *where A is a $2n \times 2n$ skew-symmetric matrix and B is an arbitrary $2n \times 2n$ matrix.*

Proof. Recalling that the determinant of a product is the product of determinants, we find that

$$\begin{aligned} \det(BAB^T) &= \det(B)\det(A)\det(B^T) \\ &= \det(B)^2 \det(A) \end{aligned}$$

Since A is skew-symmetric, it follows that BAB^T is also skew-symmetric. Using Proposition 2.2 on both A and BAB^T, we see that

$$Pf(BAB^T)^2 = det(B)^2 Pf(A)^2 \tag{2}$$

Taking the square root of both sides of Equation 2 gives

$$Pf(BAB^T) = \pm \det(B) Pf(A). \tag{3}$$

To determine the sign in Equation 3, we argue by contradiction. First note that if $Pf(A) = 0$, then we have our result, so assume $Pf(A) \neq 0$. Now, suppose $Pf(BAB^T) = -\det(B)Pf(A)$. Recall that the matrix B is arbitrary, so we can substitute the identity matrix $B = I$ into the expression. This gives $Pf(A) = -Pf(A)$, and we arrive at our contradiction. We thus conclude that $Pf(BAB^T) = \det(B)Pf(A)$. □

This identity shows us that we can do symmetric row and column operations on a matrix without changing the value of its Pfaffian. This will be a key concept in the next section.

Corollary 2.4. *Let A and B be $2n \times 2n$ skew-symmetric matrices and c be a constant such that B is obtained from A by adding c times row s to row r and c times column s to column r, where $0 \leq r < s \leq 2n$. Then $Pf(B) = Pf(A)$.*

Proof. Consider the row and column operations in terms of matrix multiplication. The row operation is equivalent to multiplying a matrix G by A where G has ones on the main diagonal and zeros everywhere else except for a single c as entry r, s. Similarly, the column operation is equivalent to multiplying A by G^T. Thus $B = GAG^T$ for our chosen G. As G is an upper-triangular matrix, it is easy to see that that $det(G) = 1$. Thus by Proposition 2.3 we see that $Pf(B) = Pf(A)$. □

Example 2.5. *Consider a matrix A*

$$A = \begin{pmatrix} 0 & 1 & 2 & 4 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -4 & -2 & -1 & 0 \end{pmatrix}.$$

Now consider another matrix B and its transpose B^T

$$B = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$B^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If we multiply B by A, we'll get the following

$$BA = \begin{pmatrix} -4 & -1 & 2 & 6 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -4 & -2 & -1 & 0 \end{pmatrix}.$$

Notice that this resulting matrix BA is equivalent to performing an elementary row operation on A, namely $R_1 + 2R_3$. A similar observation can be made for the matrix AB^T,

$$AB^T = \begin{pmatrix} 4 & 1 & 2 & 4 \\ 1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -6 & -2 & -1 & 0 \end{pmatrix}.$$

except it is equivalent to performing an elementary column operation on A, $C_1 + 2C_3$. So the matrix BAB^T,

$$BAB^T = \begin{pmatrix} 0 & -1 & 2 & 6 \\ 1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -6 & -2 & -1 & 0 \end{pmatrix}$$

is equivalent to performing those two row-column operations on A.

There is one final identity that is useful when dealing with the Pfaffian, namely the value of the Pfaffian of a matrix multiplied by a constant. It turns out that this identity is almost identical to that of the similar identity for the determinant.

Proposition 2.6. *For a $2n \times 2n$ skew-symmetric matrix A and a constant c*

$$Pf(cA) = (c^n)Pf(A)$$

Proof. Let

$$A = \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,2n} \\ -a_{1,2} & 0 & \cdots & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{2n,1} & -a_{2n,2} & \cdots & 0 \end{pmatrix}$$

If we multiply this A by a constant we'll see that

$$\begin{aligned} Pf(cA) &= Pf\left(c \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,2n} \\ -a_{1,2} & 0 & \cdots & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{2n,1} & -a_{2n,2} & \cdots & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 & ca_{1,2} & \cdots & ca_{1,2n} \\ -ca_{1,2} & 0 & \cdots & ca_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ -ca_{2n,1} & -ca_{2n,2} & \cdots & 0 \end{pmatrix} \end{aligned}$$

This gives us the following equations for the Pfaffian

$$\begin{aligned} Pf(cA) &= \sum_{\pi} sgn(\pi) ca_{i_1,j_1} \cdots ca_{i_n,j_n} \\ &= \sum_{\pi} sgn(\pi)(c^n) a_{i_1,j_1} \cdots a_{i_n,j_n} \\ &= (c^n) \sum_{\pi} sgn(\pi) a_{i_1,j_1} \cdots a_{i_n,j_n} \\ &= (c^n) Pf(A) \end{aligned}$$

□

3 The Pfaffian Transformation: Definition and Basic Facts

In the previous section we defined the Pfaffian of a skew-symmetric matrix and noted several of its properties. In this section we return to our examination of the Pfaffian transformation, beginning with a precise definition.

Definition 3.1. *The* Pfaffian transformation *of a sequence* $(a_n) = (a_0, a_1, a_2, ...)$ *is* $Pf((a_n)) = (Pf(A_0), Pf(A_1), Pf(A_2), ...)$ *where* A_n *is the* $2n+2 \times 2n+2$ *skew-symmetric matrix*

$$\begin{pmatrix} 0 & a_0 & a_1 & a_2 & \cdots & a_{2n} \\ -a_0 & 0 & a_0 & a_1 & \cdots & a_{2n-1} \\ -a_1 & -a_0 & 0 & a_0 & \cdots & a_{2n-2} \\ -a_2 & -a_1 & -a_0 & 0 & \cdots & a_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{2n} & -a_{2n-1} & -a_{2n-2} & -a_{2n-3} & \cdots & 0 \end{pmatrix}.$$

In order to facilitate discussion of individual elements of the output we also define $Pf(A_k)$ *to be* $\tilde{a}_k$*, thus we have* $Pf((a_n)) = (\tilde{a}_n) = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, ...)$.

Note that we use the notation $Pf()$ for both the Pfaffian of a matrix and the Pfaffian transformation of a sequence. This is not ambiguous so long as we are careful with our parentheses, but as an additional aid to clarity we maintain the convention of using uppercase letters for matrices and lowercase letters for scalars.

Given the relationship between the Pfaffian and the determinant of a matrix it makes sense to apply the methods of linear algebra when examining the properties of the Pfaffian transformation. In particular, the techniques of Gaussian elimination and Laplace expansion may be used with only slight modifications. We have shown via Corollary 2.4 that row and column operations such as those used in Gaussian elimination do not change the value of the Pfaffian of a matrix when they are applied symmetrically to rows and columns (e.g., adding row i to row j will not change the value of the Pfaffian so long as we also add column i to column j, whereas if we performed only one of these operations on a skew-symmetric matrix it might no longer be skew-symmetric.) We now give a useful version of Laplace expansion for the Pfaffian.

Lemma 3.2. *We have*

$$Pf\begin{pmatrix} 0 & a_{1,2} & 0 & 0 & \cdots & 0 \\ -a_{1,2} & 0 & a_{2,3} & a_{2,4} & \cdots & a_{2,2n} \\ 0 & -a_{2,3} & 0 & a_{3,4} & \cdots & a_{3,2n} \\ 0 & -a_{2,4} & -a_{3,4} & 0 & \cdots & a_{4,2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -a_{2,2n} & -a_{3,2n} & -a_{4,2n} & \cdots & 0 \end{pmatrix}$$

$$= a_{1,2}Pf\begin{pmatrix} 0 & a_{3,4} & \cdots & a_{3,2n} \\ -a_{3,4} & 0 & \cdots & a_{4,2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{3,2n} & -a_{4,2n} & \cdots & 0 \end{pmatrix}.$$

Proof. Recall from the definition of the Pfaffian that $Pf(A_n) = \sum_\pi wt(\pi)$ where the sum is over all permutations π on the set $\{1, 2, ..., 2n+2\}$ of the form

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 2n+2 \\ i_1 & j_1 & i_2 & j_2 & \cdots & j_{n+1} \end{bmatrix}$$

where $i_k < j_k$ and $i_1 < i_2 < \cdots < i_{n+1}$ and $wt(\pi)$ is a product of $sgn(\pi)$ and the matrix entries indexed by π. Observe that each product must include an entry from the first row of the matrix. Examining the above matrix, we see that the only permutations that will have non-zero weights are those that choose the second entry in the first row, as all other entries in that row are zero. Thus we may reduce the formula in this case by factoring out $a_{1,2}$, leaving us with $Pf(A_n) = a_{1,2}\sum_\pi wt(\pi)$ where the sum is over all permutations π on the set $\{3, 4, ..., 2n+2\}$ of the form

$$\pi = \begin{bmatrix} 3 & 4 & \cdots & 2n+2 \\ i_2 & j_2 & \cdots & j_{n+1} \end{bmatrix}$$

where $i_k < j_k$ and $i_2 < i_3 < \cdots < i_{n+1}$. Returning to matrix form, we have

$$a_{1,2}Pf\begin{pmatrix} 0 & a_{3,4} & \cdots & a_{3,2n} \\ -a_{3,4} & 0 & \cdots & a_{4,2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{3,2n} & -a_{4,2n} & \cdots & 0 \end{pmatrix}.$$

□

Armed with these tools we may prove specific results without returning to the definition of the Pfaffian by using row and column operations to reduce a matrix to this special case, then applying Lemma 3.2. This technique may then be extended inductively to prove general results. Towards this end, the following observation aids us by providing a trivial formula for $\tilde{a}_0$, which will make demonstrating base cases extremely simple.

Note 3.3. *For any sequence* (a_n) *we have* $\tilde{a}_0 = Pf(A_0) = Pf\begin{pmatrix} 0 & a_0 \\ -a_0 & 0 \end{pmatrix} = a_0$.

We now give a simple example of the above technique.

Proposition 3.4. $Pf((1,1,1,...)) = (1,1,1,...)$.

Proof. We argue by induction on n. For $n=0$, we have $\tilde{a}_0 = a_0 = 1$. Now suppose $n \geq 1$ and $a_{n-1} = 1$. Then $\tilde{a}_n = Pf(A_n) =$

$$Pf\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & 1 & \cdots & 1 \\ -1 & -1 & 0 & 1 & \cdots & 1 \\ -1 & -1 & -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & 0 \end{pmatrix},$$

which by subtracting row 2 from row 1 and column 1 from column 2 is equal to

$$Pf\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & 1 & \cdots & 1 \\ 0 & -1 & -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & -1 & \cdots & 0 \end{pmatrix}.$$

By Lemma 3.2 this may be reduced to

$$1 \cdot Pf\begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 0 \end{pmatrix},$$

which is simply $Pf(A_{n-1})$. Thus we have $\tilde{a}_n = Pf(A_{n-1}) = \tilde{a}_{n-1} = 1$. □

This example is actually slightly more interesting than it at first appears. In our earlier discussion of the Pfaffian we noted that $Pf(cA) = c^n Pf(A)$ where c is a scalar and A is a $2n \times 2n$ matrix. We can use this property to extend Proposition 3.4 to all constant sequences with one simple modification.

Proposition 3.5. *For any sequence (a_n) and any real number c we have $Pf((ca_n)) = (c^{n+1}\tilde{a}_n)$.*

Proof.

$$\begin{aligned} Pf((ca_n)) &= (Pf(cA_n)) \qquad \text{(by Definition 3.1)} \\ &= (c^{n+1}Pf(A_n)) \qquad \text{(by Proposition 2.6)} \\ &= (c^{n+1}\tilde{a}_n) \qquad \text{(by Definition 3.1)} \end{aligned}$$

□

Corollary 3.6. $Pf((c,c,c,...)) = (c, c^2, c^3, ...)$ *for any constant c.*

Proof. Let $(a_n) = (c,c,c,...)$.

$$\begin{aligned} Pf((a_n)) &= Pf(c(1,1,1,...)) \\ &= (c^{n+1}\tilde{a}_n) \qquad \text{(by Proposition 3.5)} \\ &= (c, c^2, c^3, ...) \qquad \text{(by Proposition 3.4)} \end{aligned}$$

□

Similarly, to determine the effects of the Pfaffian transformation on all arithmetic sequences of form $(c, 2c, 3c, ...)$ we need only consider the effect on the sequence $(1, 2, 3, ...)$.

Example 3.7. *Let* $(a_n) = (1, 2, 3, ...)$. $\tilde{a}_0 = a_0$, *thus* $\tilde{a}_0 = 1$.

Now consider A_n *for* $n \geq 1$. *Note that the difference between any two rows is simply the row vector of* $2n + 2$ *ones. Thus we see that the rows are linearly dependent, thus the determinant is 0, thus because* $Pf(A_n)^2 = \det A_n$ *it must be the case that* $\tilde{a}_n = 0$. *Therefore* $(\tilde{a}_n) = (1, 0, 0, ...)$.

We now examine a slightly more complicated sequence: the sequence of Fibonacci numbers.

Proposition 3.8. *For the sequence of Fibonacci numbers* (a_n) *where* $a_n = a_{n-1} + a_{n-2}$ *with initial conditions* $a_0 = 1, a_1 = 1$ *we have* $\tilde{a}_n = 2^n$.

Proof. We argue by induction on n. For $n = 0$, we have $\tilde{a}_0 = a_0 = 1 = 2^0$.

Now suppose $n \geq 1$ and $\tilde{a}_{n-1} = 2^{n-1}$.

We evaluate $Pf(A_n)$ by subtracting the second and third rows of A_n from the first row and the second and third columns from the first column, then apply Lemma 3.2.

$$\begin{aligned}
Pf(A_n) &= Pf\begin{pmatrix}
0 & 1 & 1 & 2 & 3 & 5 & \cdots & a_{2n} \\
-1 & 0 & 1 & 1 & 2 & 3 & \cdots & a_{2n-1} \\
-1 & -1 & 0 & 1 & 1 & 2 & \cdots & a_{2n-2} \\
-2 & -1 & -1 & 0 & 1 & 1 & \cdots & a_{2n-3} \\
-3 & -2 & -1 & -1 & 0 & 1 & \cdots & a_{2n-4} \\
-5 & -3 & -2 & -1 & -1 & 0 & \cdots & a_{2n-5} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{2n} & -a_{2n-1} & -a_{2n-2} & -a_{2n-3} & a_{2n-4} & a_{2n-5} & \cdots & 0
\end{pmatrix} \\
&= Pf\begin{pmatrix}
0 & 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-2 & 0 & 1 & 1 & 2 & 3 & \cdots & a_{2n-1} \\
0 & -1 & 0 & 1 & 1 & 2 & \cdots & a_{2n-2} \\
0 & -1 & -1 & 0 & 1 & 1 & \cdots & a_{2n-3} \\
0 & -2 & -1 & -1 & 0 & 1 & \cdots & a_{2n-4} \\
0 & -3 & -2 & -1 & -1 & 0 & \cdots & a_{2n-5} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -a_{2n-1} & -a_{2n-2} & -a_{2n-3} & a_{2n-4} & a_{2n-5} & \cdots & 0
\end{pmatrix} \\
&= 2Pf\begin{pmatrix}
0 & 1 & 1 & 2 & \cdots & a_{2n-2} \\
-1 & 0 & 1 & 1 & \cdots & a_{2n-3} \\
-1 & -1 & 0 & 1 & \cdots & a_{2n-3} \\
-2 & -1 & -1 & 0 & \cdots & a_{2n-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{2n-2} & -a_{2n-3} & a_{2n-4} & a_{2n-5} & \cdots & 0
\end{pmatrix} \\
&= 2Pf(A_{n-1}).
\end{aligned}$$

Thus we have $\tilde{a}_n = 2Pf(A_{n-1}) = 2\tilde{a}_{n-1} = 2(2^{n-1}) = 2^n$ by induction. □

In fact, we may extend this result to cover a more general class of sequences which satisfy two-term linear recurrence relations.

Proposition 3.9. *Let α and β be real numbers. For any sequence (a_n) where $a_n = \alpha a_{n-1} + \beta a_{n-2}$ for $n \geq 3$ with initial conditions $a_1 = 1$ and $a_2 = \alpha$ we have $\tilde{a}_n = (1+\beta)^n$.*

Proof. We argue by induction on n. For $n = 0$, we have $\tilde{a}_0 = 1 = (1+\beta)^0$.

Now suppose $n \geq 1$ and $a_{n-1} = (1+\beta)^{n-1}$.

To evaluate $Pf(A_n)$ we first subtract α times the second row and β times the third row from the first row, then subtract α times the second column and β times the third column from the first column, then apply Lemma 3.2.

$$\begin{aligned}
Pf(A_n) &= Pf\begin{pmatrix} 0 & 1 & \alpha & \alpha^2+\beta & \cdots & a_{2n} \\ -1 & 0 & 1 & \alpha & \cdots & a_{2n-1} \\ -\alpha & -1 & 0 & 1 & \cdots & a_{2n-2} \\ -(\alpha^2+\beta) & -\alpha & -1 & 0 & \cdots & a_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{2n} & -a_{2n-1} & -a_{2n-2} & -a_{2n-3} & \cdots & 0 \end{pmatrix} \\
&= Pf\begin{pmatrix} 0 & 1+\beta & 0 & 0 & \cdots & 0 \\ -(1+\beta) & 0 & 1 & \alpha & \cdots & a_{2n-1} \\ 0 & -1 & 0 & 1 & \cdots & a_{2n-2} \\ 0 & -\alpha & -1 & 0 & \cdots & a_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -a_{2n-1} & -a_{2n-2} & -a_{2n-3} & \cdots & 0 \end{pmatrix} \\
&= (1+\beta)Pf\begin{pmatrix} 0 & 1 & \cdots & a_{2n-2} \\ -1 & 0 & \cdots & a_{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{2n-2} & -a_{2n-3} & \cdots & 0 \end{pmatrix} \\
&= (1+\beta)Pf(A_{n-1}).
\end{aligned}$$

Thus we have $\tilde{a}_n = (1+\beta)Pf(A_{n-1}) = (1+\beta)\tilde{a}_{n-1} = (1+\beta)(1+\beta)^{n-1} = (1+\beta)^n$. □

Note that this result includes not only Proposition 3.8 but also Corollary 3.6 and Example 3.7 as special cases. (To apply Proposition 3.9 to Example 3.7, consider the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$). In the next section, we will show how this technique of using the linear recurrence of an input sequence to reduce the number of non-zero terms can be extended to create an algorithm that may be applied to sequences with linear recurrences of any (finite) length.

4 A Reduction

In this section, we introduce a reduction in the number of permutations that need to be considered in evaluating a Pfaffian image. This algorithm is based on reducing a matrix using elementary row and column operations. As a first step, note that the definition of the Pfaffian transformation on sequences places a constraint on the type of matrix that are to be considered. Specifically, the definition of the Pfaffian transformation constructs a skew-symmetric Toeplitz matrix from a sequence.

Definition 4.1. *Let* $\{a_k\}$ *be a set of* $2n-1$ *numbers, where* $k = -n+1, ..., -1, 0, 1, ..., n-1$*; a* Toeplitz matrix A *is an* $n \times n$ *matrix such that* $A_{ij} = a_{i-j}$*.*

Definition 4.2. A banded matrix *is an* $n \times n$ *matrix* A *such that given some* $m \leq n$*, the entry* $A_{ij} = 0$ *for* $|i - j| > m$*.*

We now state a generalized form of the reduction.

Proposition 4.3. *Let* A *be a skew-symmetric Toeplitz matrix whose entries* $(a_0, a_1, a_2, ...)$ *satisfy an* N*-term linear recurrence relation with constant coefficients, given by* $a_i = \alpha_1 a_{i-1} + \alpha_2 a_{i-2} + ... + \alpha_N a_{i-N}$*; consider the following algorithm,*

1.) Perform row subtractions by replacing each A_{ij} *with* $B'_{ij} = A_{ij} - \sum_{k=1}^{k=N} \alpha_k A_{(i+k)j}$

2.) Perform column subtractions by replacing each B'_{ij} *with* $B_{ij} = B'_{ij} - \sum_{l=1}^{l=N} \alpha_l B'_{i(j+l)}$

This algorithm reduces A *to a banded skew-symmetric matrix* B *with at most* N *nonzero bands and with homogenous entries on each band, except at a submatrix at the bottom-right corner of size at most* $N \times N$*.*

Proof. Let $(a_0, a_1, a_2, ...)$ be a sequence satisfying an N-term recurrence relation given by $a_i = \alpha_1 a_{i-1} + ... + \alpha_N a_{i-N}$. Consider the $n \times n$ skew-symmetric Toeplitz matrix A defined on this sequence:

$$A = \begin{pmatrix} 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n \\ -a_0 & 0 & a_0 & \cdots & a_{n-2} & a_{n-1} \\ -a_1 & -a_0 & 0 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & 0 & a_0 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_0 & 0 \end{pmatrix} \tag{4}$$

By Corollary 2.4, $Pf(B) = Pf(A)$. The matrix B can be inspected in two parts to ensure that it is indeed of a banded skew-symmetric form.

Case 1: Consider the upper triangle of B, namely all the entries of the matrix B above its main diagonal. Step 1.) assures that all entries above the N^{th} diagonal vanish, since the N-term recurrence relation gives $a_i - \sum_{k=1}^{N} \alpha_k a_{i-k} = 0$. B thus has a banded upper triangle. Since the algorithm performs the same operations on all entries in a given band except for those entries that are N rows from the last row or N columns away from the last column, the bands are homogenous with the exception of a submatrix in the bottom-right corner of size at most $N \times N$.

Case 2: Consider the lower triangle of B, namely all the entries of the matrix B below its main diagonal. Note that the reduction algorithm is equivalent to matrix multiplying A from the left by the matrix G, and from the right by the matrix G^T where G is of the form:

$$G = \begin{pmatrix} 1 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & \cdots \\ 0 & 1 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots \\ 0 & 0 & 1 & -\alpha_1 & -\alpha_2 & \cdots \\ 0 & 0 & 0 & 1 & -\alpha_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{5}$$

It is known from Case 1 that $B = GAG^T$ has banded upper triangle. Consider B^T.

$$B^T = (GAG^T)^T = G^T A^T G = -GAG^T = -B \tag{6}$$

Equation (6) implies that B has a banded lower triangle, assuring that all entries below the N^{th} diagonal vanish. Equation (6) further implies that B is skew-symmetric. Thus, the matrix resulting from applying the reduction algorithm on a skew-symmetric Toeplitz matrix results in a banded skew-symmetric matrix with at most N nonzero diagonals. ☐

Proposition 4.3 assures that a skew-symmetric Toeplitz matrix A whose entries satisfy a linear recurrence relation can be reduced to a banded skew-symmetric matrix B, with homogenous banded entries except at the bottom-right corner, and with $Pf(B) = Pf(A)$. Thus, given a sequence $(a_1, a_2, ..., a_n, ...)$ that satisfies a linear recurrence relation, there always exists another sequence $(x_1, ..., x_N, 0, 0, ...)$ with a finite number of nonzero terms, and whose Pfaffian image sequence satisfies the same linear recurrence relation. From this point onward, we will study the action of the Pfaffian transformation on reduced sequences of the form $(x_1, ..., x_N, 0, 0, ...)$, in place of their pre-reduction counterparts.

A stronger statement can be made about the values of $x_1, ..., x_N$, for the class of sequences whose initial N terms also satisfy its linear recurrence relation. That is, given a recurrence relation $a_i = \sum_{k=1}^{N} \alpha_k a_{i-k}$, the first N terms are fixed as $a_i = \sum_{k=1}^{i} \alpha_i a_{i-k}$ for $n \leq N$. An example of this was shown in Proposition 3.9, where the skew-symmetric Toeplitz matrix generated from a sequence that satisfies a general 2-term linear recurrence proved to be of a homogenous striped form, given that the 2 initial entries are of a particular form. Since the Pfaffian of the $N = 1$ case has been exactly solved in Corollary 3.6, we consider the reduction of cases where $N \geq 2$.

Proposition 4.4. *Let $(a_0, a_1, a_2, ...)$ be a sequence satisfying an N-term recurrence relation given by $a_i = \sum_{k=1}^{N} \alpha_k a_{i-k}$, with the N initial terms fixed to satisfy the recurrence relation*

$$\begin{aligned} a_0 &= 1 \\ a_1 &= \alpha_1 \\ a_2 &= \alpha_1^2 + \alpha_2 \end{aligned}$$

$$\vdots$$

$$a_N = \sum_{k=1}^{k=N} \alpha_k a_{N-k}$$

Then $Pf(a_0, a_1, a_2, ...) = Pf(\alpha_2 + 1, \alpha_3, ..., \alpha_N, 0, 0, ...)$

Proof. Consider the skew-symmetric Toeplitz matrix A formed from the sequence $(a_0, a_1, a_2, ...)$, given by

$$A = \begin{pmatrix} 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n \\ -a_0 & 0 & a_0 & \cdots & a_{n-2} & a_{n-1} \\ -a_1 & -a_0 & 0 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & 0 & a_0 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_0 & 0 \end{pmatrix} \tag{7}$$

Applying the reduction algorithm described in Proposition 4.3 produces a reduced matrix B. The upper triangle of B can be inspected in four parts to ensure that it is of a banded form.

Case 1: Consider an entry $A_{qr} = a_i$ on the $(i+1)^{th}$ diagonal above the main diagonal, for $i > N-2$. The reduction algorithm described in Proposition 4.3 produces a matrix B such that the corresponding entry in the reduced matrix is clearly $B_{qr} = 0$.

Case 2: Consider an entry $A_{qr} = a_i$ on the $(i+1)^{th}$ diagonal above the main diagonal, for $0 < i \leq N-2$. Let S be the $(N+1) \times (N+1)$ submatrix of A that $S_{11} = A_{qr}$

$$S = \begin{pmatrix} a_i & a_{i+1} & a_{i+2} & a_{i+3} & a_{i+4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ a_0 & a_1 & a_2 & a_3 & a_4 & \ddots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ -a_0 & 0 & a_0 & a_1 & a_2 & \cdots \\ -a_1 & -a_0 & 0 & a_0 & a_1 & \cdots \\ -a_2 & -a_1 & -a_0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{8}$$

Let m index the rows and n index the columns of this submatrix S, such that the main diagonal entries of A correspond to the entries S_{mn} for $m = n + i + 1$. Define the upper triangle region as the entries above this main diagonal, and denote the lower triangle region as the entries below this main diagonal.

For entries in the first row, given the constraint $m < n + i + 1$ that restricts S_{mn} to the upper triangle region, the recurrence relation dictates that

$$S_{1n} = \sum_{1<m<n+i+1} \alpha_{m-1} S_{mn} \tag{9}$$

For entries in the first column, given the constraint $n < m - i - 1$ that restricts S_{mn} to the lower triangle region, the recurrence relation again dictates that

$$S_{m1} = \begin{cases} \sum_{1<n<m-i-1} \alpha_{n-1} S_{mn} & \text{for } m > i+3 \\ a_0 & \text{for } m = i+3 \end{cases} \tag{10}$$

Note that choosing N initial terms that also satisfy the recurrence relation is critical in ensuring that Equation (9) and Equation (10) hold for entries close to the main diagonal, since the entry values in this region are determined by the initial terms.

Consider the reduction algorithm described in Proposition 4.3, and its effect on the S_{11} entry. Step 1.) in this algorithm subtracts by row, so it follows from Equation (9) that subtracting the upper triangle entries below S_{11} results in an exact cancellation. It then follows from Equation (10) that the corresponding lower triangle entries below S_{1l} are the only surviving contribution. The resultant matrix S' then has the entry S'_{11} given by

$$S'_{11} = \sum_{i+2<m<N+1} \alpha_{m-1} \left(\sum_{1<n<m-i-1} \alpha_{n-1} S_{mn} \right) = \alpha_{i+2} + \left(\sum_{i+3<m<N+1} \alpha_{m-1} \sum_{1<n<m-i-1} \alpha_{n-1} S_{mn} \right) \tag{11}$$

Note that for a $l > 1$, the other entries on the first row of S' are given by

$$S'_{1l} = \sum_{l+i+1<m<N+1} \alpha_{m-1} S_{ml} \tag{12}$$

Step 2.) in the reduction algorithm subtracts by column, so it follows that the resulting S'' matrix has the entry S''_{11} given by

$$S''_{11} = S'_{11} - \sum_{l \leq n<m-i-1} \alpha_{n-1} \left(\sum_{l+i+1<m<N+1} \alpha_{m-1} S_{ml} \right) \tag{13}$$

Noting that $l > 1$, and using the expression for S'_{11} given by Equation (11), this can be reexpressed

$$S''_{11} = \left(\alpha_{i+2} + \sum_{i+3<m<N+1} \alpha_{m-1} \sum_{1<n<m-i-1} \alpha_{n-1} S_{mn} \right) - \left(\sum_{1<n<m-i-1} \alpha_{n-1} \sum_{i+3<m<N+1} \alpha_{m-1} S_{mn} \right) \tag{14}$$

Interchanging the summations over m and n, the double summations exactly cancel, leaving the α_{i+2} term.

(It is important to note that if S is a submatrix at the bottom-right of A such that S is $M \times M$ for $M < N+1$, the result still holds due to the form of the initial conditions of the sequence.)

Thus, given the entry in the initial matrix $A_{qr} = a_i$ for $0 < i < N$, the corresponding entry in the reduced matrix is $B_{qr} = \alpha_{i+2}$.

Case 3: Consider an entry on the diagonal immediately above the main diagonal, such that $A_{qr} = a_i$ for $i = 0$. Define the submatrix S analogous to that in Case 2

$$S = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ -a_0 & 0 & a_0 & a_1 & a_2 & \cdots \\ -a_1 & -a_0 & 0 & a_0 & a_1 & \cdots \\ -a_2 & -a_1 & -a_0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{15}$$

In this case, the $S_{11} = a_0 = 1$ entry at the top-left corner of S is not cancelled by any upper triangle entries, since there are no such entries below S_{11}. There is then an extra 1 added to the result from Case 2. Thus, given the entry in the initial matrix $A_{qr} = a_0$, the corresponding entry in the reduced matrix is $B_{qr} = \alpha_2 + 1$.

Case 4: Consider the N^{th} column of A. Since there are no lower triangle entries below the main diagonal for this last column of A, the reduction algorithm described in Proposition 4.3 produces entries of 0, except for $A_{n-1,n} = A_{n,n-1} = 1$.

Let $x_1 = \alpha_2 + 1, x_2 = \alpha_3, ..., x_{N-1} = \alpha_N$. These cases show that the reduction algorithm produces a matrix with an upper triangle of the form

$$\begin{pmatrix} * & x_1 & x_2 & \cdots & x_{N-1} & \cdots & 0 & 0 \\ * & * & x_1 & \cdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & x_2 & \cdots & x_{N-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & * & x_1 & x_2 & 0 \\ * & * & * & \cdots & * & * & x_1 & 0 \\ * & * & * & \cdots & * & * & * & 1 \\ * & * & * & \cdots & * & * & * & * \end{pmatrix} \tag{16}$$

By Proposition 4.3, it is then known that the whole matrix has the form:

$$\begin{pmatrix} 0 & x_1 & x_2 & \cdots & x_{N-1} & \cdots & 0 & 0 \\ x_1 & 0 & x_1 & \cdots & \vdots & \ddots & \vdots & \vdots \\ x_2 & x_1 & 0 & \cdots & x_2 & \cdots & x_{N-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ x_{N-1} & \cdots & x_2 & \cdots & 0 & x_1 & x_2 & 0 \\ \vdots & \ddots & \vdots & \ddots & x_1 & 0 & x_1 & 0 \\ 0 & \cdots & x_{N-1} & \cdots & x_2 & x_1 & 0 & 1 \\ 0 & \cdots & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix} \tag{17}$$

By Lemma 3.2, this matrix can be further reduced to a banded skew-symmetric matrix

B of the form

$$B = \begin{pmatrix} 0 & x_1 & x_2 & \cdots & x_{N-1} & \cdots & 0 \\ x_1 & 0 & x_1 & \cdots & \vdots & \ddots & \vdots \\ x_2 & x_1 & 0 & \cdots & x_2 & \cdots & x_{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{N-1} & \cdots & x_2 & \cdots & 0 & x_1 & x_2 \\ \vdots & \ddots & \vdots & \ddots & x_1 & 0 & x_1 \\ 0 & \cdots & x_{N-1} & \cdots & x_2 & x_1 & 0 \end{pmatrix} \tag{18}$$

Thus, comparing (17) and (28), $Pf(a_0, a_1, a_2, ...) = Pf(x_1, x_2, ..., x_{N-1}, 0, 0, ...) = Pf(\alpha_2 + 1, \alpha_3, ..., \alpha_N, 0, 0, ...)$. □

Proposition 4.4 is useful in practice, as it supplies us with the values of $x_1, x_2, ..., x_{N-1}$, the nonzero terms in the reduced sequence (in fact, compared to the general case described in Proposition 4.3, we have even succeeded in decreasing the number of nonzero terms by one). In the following sections, we will refer to either Proposition 4.3 or Proposition 4.4, depending on the desired level of generality.

5 An Algebraic Approach

Armed with the results of the previous section, we see that we may find the recurrence relation satisfied by the Pfaffian transformation of any sequence that satisfies an N-term linear recurrence by understanding the effects of the transformation on a sequence of form $(x_1, x_2, ..., x_N, 0, 0, ...)$. We thus use the linear algebraic methods of Section 3 to examine such sequences, beginning with the simplest.

Proposition 5.1. $Pf((a, 0, 0, ...)) = (a^{n+1})$

Proof. We argue by induction on n. For $n = 0$, we have $\tilde{a}_0 = a_0 = a = a^{0+1}$.

Now suppose $n \geq 1$ and $a_{n-1} = a^n$.

We have

$$\begin{aligned} Pf(A_n) &= Pf \begin{pmatrix} 0 & a & 0 & 0 & \cdots & 0 \\ -a & 0 & a & 0 & \cdots & 0 \\ 0 & -a & 0 & a & \cdots & 0 \\ 0 & 0 & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\ &= a \cdot Pf \begin{pmatrix} 0 & a & \cdots & 0 \\ -a & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\ &= aPf(A_{n-1}) = a(a^n) = a^{n+1} \end{aligned}$$

Thus we have $\tilde{a}_n = a^{n+1}$ by induction. □

Proposition 5.2. *Let* $(a_n) = (a, b, 0, 0, ...)$. *Then* $(\tilde{a}_n)$ *satisfies the linear recurrence* $\tilde{a}_n = a \cdot a_{n-1} - b^2 \cdot a_{n-2}$.

Proof.

$$Pf(A_n) = Pf \begin{pmatrix} 0 & a & b & 0 & 0 & 0 & \cdots & 0 \\ -a & 0 & a & b & 0 & 0 & \cdots & 0 \\ -b & -a & 0 & a & b & 0 & \cdots & 0 \\ 0 & -b & -a & 0 & a & b & \cdots & 0 \\ 0 & 0 & -b & -a & 0 & a & \cdots & 0 \\ 0 & 0 & 0 & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

This matrix can be evaluated via a set of repeated row/column operations starting in the upper-left corner.

First, let c_0 be the first entry in the second column.

Then subtract the second column multiplied by b/c_0 from the third column and subtract the second row multiplied by b/c_0 from the third row.

Finally, expand along the first row/column using Lemma 3.2.

This sequence of operations will give us c_0 multiplied by the Pfaffian of a new matrix with 2 fewer rows and 2 fewer columns that is otherwise identical to the previous matrix except perhaps for a new value for the first entry in the second column, which we will call c_1, so $c_1 = a - b^2/c_0$ (and, naturally, the first entry in the second row, which will simply be $-c_1$).

Base case:

$$\begin{aligned}
Pf(A_n) &= Pf\begin{pmatrix} 0 & a & b & 0 & 0 & 0 & \cdots & 0 \\ -a & 0 & a & b & 0 & 0 & \cdots & 0 \\ -b & -a & 0 & a & b & 0 & \cdots & 0 \\ 0 & -b & -a & 0 & a & b & \cdots & 0 \\ 0 & 0 & -b & -a & 0 & a & \cdots & 0 \\ 0 & 0 & 0 & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\
&= Pf\begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 & \cdots & 0 \\ -a & 0 & a & b & 0 & 0 & \cdots & 0 \\ 0 & -a & 0 & a-b^2/a & b & 0 & \cdots & 0 \\ 0 & -b & -a+b^2/a & 0 & a & b & \cdots & 0 \\ 0 & 0 & -b & -a & 0 & a & \cdots & 0 \\ 0 & 0 & 0 & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\
&= aPf\begin{pmatrix} 0 & a-b/a & b & 0 & \cdots & 0 \\ -a+b/a & 0 & a & b & \cdots & 0 \\ -b & -a & 0 & a & \cdots & 0 \\ 0 & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}
\end{aligned}$$

And in general:

$$
\begin{aligned}
Pf(A_n) &= Pf\begin{pmatrix} 0 & c_k & b & 0 & 0 & 0 & \cdots & 0 \\ -c_k & 0 & a & b & 0 & 0 & \cdots & 0 \\ -b & -a & 0 & a & b & 0 & \cdots & 0 \\ 0 & -b & -a & 0 & a & b & \cdots & 0 \\ 0 & 0 & -b & -a & 0 & a & \cdots & 0 \\ 0 & 0 & 0 & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\
&= Pf\begin{pmatrix} 0 & c_k & 0 & 0 & 0 & 0 & \cdots & 0 \\ -c_k & 0 & a & b & 0 & 0 & \cdots & 0 \\ 0 & -a & 0 & a-b^2/c_k & b & 0 & \cdots & 0 \\ 0 & -b & -a+b^2/c_k & 0 & a & b & \cdots & 0 \\ 0 & 0 & -b & -a & 0 & a & \cdots & 0 \\ 0 & 0 & 0 & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\
&= c_k Pf\begin{pmatrix} 0 & a-b^2/c_k & b & 0 & \cdots & 0 \\ -a+b^2/c_k & 0 & a & b & \cdots & 0 \\ -b & -a & 0 & a & \cdots & 0 \\ 0 & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}
\end{aligned}
$$

Thus by induction we see that $Pf(A_n) = c_0c_1...c_n$. Thus $\tilde{a_n} = c_0c_1...c_n$.

Note that $c_0 = a$, $c_1 = a - b^2/a$, and in general $c_{n+1} = a - b^2/c_n$. Let $(c_n) = (u_n/l_n)$. Thus we have $u_{n+1}/l_{n+1} = c_{n+1} = a - b^2/c_n = a - l_nb^2/u_n = (au_n - l_nb^2)/u_n$. Thus $l_{n+1} = u_n$, thus the product $\tilde{a}_n = c_0c_1...c_n$ telescopes to $\tilde{a}_n = u_n = au_{n-1} - l_{n-1}b^2 = au_{n-1} - b^2u_{n-2}$. Thus $(\tilde{a}_n)$ satisfies the linear recurrence $\tilde{a}_n = aa_{n-1} - b^2a_{n-2}$ for $n \geq 2$. $\square$

It should be emphasized that the above proof's reliance on division means that it does not hold for all cases–specifically, it does not hold for cases where the algorithm would involve dividing by zero. (We will later show that the result still holds in such cases.)

Proposition 5.3. *Let* $(a_n) = (a, b, c, 0, 0, ...)$. *Then* $(\tilde{a}_n)$ *satisfies the linear recurrence* $\tilde{a}_n = (a-c)\tilde{a}_{n-1} + (2ac - b^2)\tilde{a}_{n-2} + (ac^2 - c^3)\tilde{a}_{n-3} - (c^4)\tilde{a}_{n-4}$ *for* $n \geq 4$.

Proof.

$$Pf(A_n) \quad = \quad Pf \begin{pmatrix} 0 & a & b & c & 0 & 0 & \cdots & 0 \\ -a & 0 & a & b & c & 0 & \cdots & 0 \\ -b & -a & 0 & a & b & c & \cdots & 0 \\ -c & -b & -a & 0 & a & b & \cdots & 0 \\ 0 & -c & -b & -a & 0 & a & \cdots & 0 \\ 0 & 0 & -c & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

This matrix can be evaluated via a set of row/column operations and Lemma 3.2.

First, let X_0 be the first entry in the second column, Y_0 be the first entry in the third column, and Z_0 be the second entry in the third column.

Subtract the second column multiplied by $-Y_0/X_0$ from the third column and subtract the second row multiplied by $-Y_0/X_0$ from the third row.

Subtract the second column multiplied by $-c/X_0$ from the fourth column and subtract the second row multiplied by $-c/X_0$ from the fourth row.

Then expand using Lemma 3.2.

This sequence of operations will give us x_0 multiplied by a new matrix with 2 fewer rows and 2 fewer columns that is otherwise identical to the previous matrix except perhaps in 6 places. Let X_1 be the first entry in the second column, Y_1 be the first entry in the third column, and Z_1 be the second entry in the third column. (Naturally, the second entry in the first column will be $-X_1$ and so on, as the matrix remains skew-symmetric.) These operations may then be repeated, incrementing the subscripts by one.

In general:

$$
\begin{aligned}
& Pf\begin{pmatrix} 0 & X_k & Y_k & c & 0 & 0 & \cdots & 0 \\ -X_k & 0 & Z_k & b & c & 0 & \cdots & 0 \\ -Y_k & -Z_k & 0 & a & b & c & \cdots & 0 \\ -c & -b & -a & 0 & a & b & \cdots & 0 \\ 0 & -c & -b & -a & 0 & a & \cdots & 0 \\ 0 & 0 & -c & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\
= \quad & Pf\begin{pmatrix} 0 & X_k & 0 & 0 & 0 & 0 & \cdots & 0 \\ -X_k & 0 & Z_k & b & c & 0 & \cdots & 0 \\ 0 & -Z_k & 0 & X_{k+1} & Y_{k+1} & c & \cdots & 0 \\ 0 & -b & -X_{k+1} & 0 & Z_{k+1} & b & \cdots & 0 \\ 0 & -c & -Y_{k+1} & -Z_{k+1} & 0 & a & \cdots & 0 \\ 0 & 0 & -c & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\
= \quad & X_k Pf\begin{pmatrix} 0 & X_{k+1} & X_{k+1} & c & \cdots & 0 \\ -X_{k+1} & 0 & Z_{k+1} & b & \cdots & 0 \\ -Y_{k+1} & -Z_{k+1} & 0 & a & \cdots & 0 \\ -c & -b & -a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}
\end{aligned}
$$

Thus by induction $\tilde{a}_n = X_0 X_1 ... X_n$, where we have the following equations:

$$
\begin{aligned}
X_n &= a + cZ_{n-1}/X_{n-1} - bY_{n-1}/X_{n-1} \\
Y_n &= b - cY_{n-1}/X_{n-1} \\
Z_n &= a - c^2/X_{n-1}
\end{aligned}
$$

Note that

$$
\begin{aligned}
cX_{n-1} - bY_{n-1} &= cX_{n-1} - b^2 + bcY_{n-2}/X_{n-2} \\
&= c[a + cZ_{n-2}/X_{n-2} - bY_{n-2}/Xn-2] - b^2 + bcY_{n-2}/X_{n-2} \\
&= c[a + cZ_{n-2}/X_{n-2}] - b^2 \\
&= ac + c^2 Z_{n-2}/X_{n-2} - b^2
\end{aligned}
$$

Thus

$$\begin{aligned}
X_n &= (a-c) + cZ_{n-1}/X_{n-1} + (cX_{n-1} - bY_{n-1})/Xn - 1 \\
&= (a-c) + cZ_{n-1}/X_{n-1} + (ac + c^2 Z_{n-2}/X_{n-2} - b^2)/X_{n-1} \\
&= (a-c) + c[a - c^2/X_{n-2}]/X_{n-1} + (ac + c^2[a - c^2/X_{n-3}]/X_{n-2} - b^2)/X_{n-1} \\
&= (a-c) + c[a - c^2/X_{n-2}]/X_{n-1} + (ac + c^2[a - c^2/X_{n-3}]/X_{n-2} - b^2)/X_{n-1} \\
&= (a-c) + [ac - c^3/X_{n-2}]/X_{n-1} + (ac + [ac^2 - c^4/X_{n-3}]/X_{n-2} - b^2)/X_{n-1} \\
&= (a-c) + (ac + ac - b^2)/x_{n-1} + (ac^2 - c^3)/(X_{n-1}X_{n-2}) - c^4/(X_{n-1}X_{n-2}X_{n-3}) \\
&= (a-c) + (2ac - b^2)/x_{n-1} + (ac^2 - c^3)/(X_{n-1}X_{n-2}) - c^4/(X_{n-1}X_{n-2}X_{n-3})
\end{aligned}$$

Thus

$$\begin{aligned}
\tilde{a}_n &= [X_0 ... X_{n-1}][X_n] \\
&= [X_0 ... X_{n-1}]\left[(a-c) + \frac{2ac - b^2}{X_{n-1}} + \frac{ac^2 - c^3}{X_{n-1}X_{n-2}} + \frac{c^4}{X_{n-1}X_{n-2}X_{n-3}}\right] \\
&= (a-c)[X_0 ... X_{n-1}] + (2ac - b^2)[X_0 ... X_{n-2}] + (ac^2 - c^3)[X_0 ... X_{n-3}] + c^4[X_0 ... X_{n-4}] \\
&= (a-c)\tilde{a}_{n-1} + (2ac - b^2)\tilde{a}_{n-2} + (ac^2 - c^3)\tilde{a}_{n-3} - (c^4)\tilde{a}_{n-4}
\end{aligned}$$

□

It should be noted that this proof again has the disadvantage of relying on division, invalidating it in cases where the algorithm would lead to division by zero. (As with the previous proposition, we will later show that this formula still holds in such a case.) It should also be noted that this algorithm is becoming exponentially more complex as we increase the number of non-zero terms in the sequence we are examining. In the next section we present a graph-theoretic approach that conquers both of these difficulties.

6 A Graph-Theoretic Approach

In this section, we demonstrate that the Pfaffian can be evaluated solely from graph-theoretic considerations. We introduce key concepts, which we will extend to serve as a foundation for this paper's final result, as presented in the next section.

In section 2 we defined the Pfaffian of a $2n \times 2n$ skew-symmetric matrix as a sum over a certain class of permutations of $\{1, 2, ..., 2n\}$. In this section we examine the properties of these permutations more fully and prove that there exists a bijection between these permutations and the perfect matchings on the complete graph K_{2n}.

Let i denote the i^{th} vertex of K_n. Consider the most obvious perfect matching on K_{2n} - where 1 is adjacent to 2, 3 to 4, and so on. Notice that we can represent this perfect matching as an ordering of the vertices $1, \ldots, 2n$, namely $1, 2, 3, \ldots, 2n-2, 2n-1, 2n$. Consider what happens when we switch the first and second elements of our ordering. In terms of our perfect matching, nothing changes. This is true whenever we switch the j^{th} with the $(j+1)^{th}$ element of our ordering given that j is odd. Also notice that when we switch the first and second elements with the third and fourth elements of our ordering, nothing is changed with the perfect matching. This is generally true whenever we switch a set of two adjacent elements with two other adjacent elements as long as the first element of each set is in an odd position in our original ordering.

These observations lead to the following definition.

Definition 6.1. *Let* π *be a permutation on* $1, 2, \ldots, 2n$*. Then let* $H_\pi = \{\{\pi(2i-1), \pi(2i)\}|1 \leq i \leq 2n\}$*. We consider two permutations,* π_j *and* π_k*, to be* PM-equivalent *if and only if* $H_{\pi_j} = H_{\pi_k}$*.*

Example 6.2. (56783412)*,* (34127856)*,* (34561278) *are all PM-equivalent to* (12345678)*.*

We can see that for every equivalence class, there are $2^n n!$ permutations: there are n sets in each set H and we can order each one 2 different ways. Notice that in each equivalence class we have one permutation with the following ordering.

Definition 6.3. *A* Pfaffian Permutation*,* π*, of* $1, 2, \ldots, 2n$ *is a permutation of* $1, 2, \ldots, 2n$ *where* $\pi(2i-1) < \pi(2i)$*, for* $1 \leq i \leq n$ *and* $\pi(2j-1) < \pi(2k-1)$ *for* $1 \leq j, k \leq n$ *and* $j < k$*.*

Clearly we have exactly one Pfaffian Permutation in each equivalence class. Now we can consider the explicit relationship between the perfect matchings of K_{2n} and the set of our special Pfaffian Permutations. In particular we can define a bijection between the two sets.

Theorem 6.4. *There exists a bijection, f, between the set of perfect matchings of* K_{2n} *and the set of Pfaffian Permutations of* $1, 2, \ldots, 2n$*.*

Proof. Let f be the map from the perfect matchings on K_{2n} to the set of Pfaffian Permutations. Just as we did in the earlier paragraph, we assume the vertices of K_{2n} are ordered $1, 2, \ldots, 2n$. We define f explicitly as the function that follows this algorithm:

1. Consider a perfect matching, P, of K_{2n}. Let A be the set of vertices of K_{2n}. Then A is $\{1, 2, \ldots, 2n\}$. Let π be an empty list.

2. Remove i from A, where i is the smallest element of A. Append i to π.

3. Remove j from A, where j is the vertex adjacent to i in P. Append j to π.

4. Repeat steps 2 and 3 until A is exhausted.

Notice that step 2 appends an element to π in the odd positions and that step 3 appends an element in the even positions. Because for each iteration of step 2 we are choosing the minimum integer from A we see that $\pi(2i-1) < \pi(2i)$ for $1 \leq i \leq n$ and that $\pi(2j-1) < \pi(2k-1)$ for $1 \leq j, k, \leq n$ and $j < k$. Then π is a Pfaffian Permutation.

Consider the inverse of f. If we start with a Pfaffian Permutation, π, we can create a perfect matching simply by considering each $\pi(2i-1)$ and $\pi(2i)$ for $1 \leq i \leq n$ as a pair of adjacent vertices in our perfect matching. Thus f is a bijection. □

The previous theorem leads us to the following fact: The number of perfect matchings on K_{2n} is the same as the number of Pfaffian Permutations of $1, 2, \ldots, 2n$.

Theorem 6.5. *There are* $(2n-1)!! = (2n-1)(2n-3)\ldots 3 \cdot 1$ *Pfaffian Permutations of a set of distinct integers of size* $2n$

Proof. Consider the number of PM-equivalence classes for permutations of $1, 2, \ldots, 2n$. We know there are $2n!$ total permutations and $2^n n!$ permutations in each equivalence class. Then the number of equivalence classes is

$$\frac{2n!}{2^n n!} = \frac{2n!}{(2n)(2n-2)(2n-4)\ldots 4 \cdot 2} = (2n-1)(2n-3)(2n-5)\ldots 5 \cdot 3 \cdot 1 \qquad (19)$$

Then there are $(2n-1)!!$ equivalence classes and so there are $(2n-1)!!$ Pfaffian Permutations. □

Corollary 6.6. *For a complete graph on* $2n$ *vertices, there are* $(2n-1)!!$ *perfect matchings.*

Proof. The result is a direct consequence of Theorems 3.4 and 3.6. □

Recall that given a sequence $(a_1, a_2, ..., a_n, ...)$, we can construct a skew-symmetric Toeplitz matrix A as discussed in Section 2. This matrix A can then be reduced to a banded matrix B with diagonals given by the sequence with a finite number of nonzero terms $(x_1, ..., x_N)$, as discussed in Section 5. We begin by defining a graph whose edge weights can be interpreted as the entry values of this matrix B. We use the idea of a Pfaffian Permutation to give an ordering of the vertices of this graph.

Definition 6.7. *Fix* $r \geq 1$ *and* $n \geq 1$*. The* r-claw *on* n *vertices is the graph* $C^r_{2n} = (V, E)$ *with vertices* $1, 2, \ldots, n$*, such that vertices* $i \in V$ *and* $j \in V$ *are connected with an edge* $ij \in E$*, whenever* $|i - j| \leq r$*.*

Note that in accordance with our definition of the Pfaffian Permutation, the vertices of this graph are ordered from $1, 2, ..., n$ in ascending order. At this point, it is important to stress the connection between claw graphs and banded matrices. Note that for such a matrix B, there exists $r \in \mathbb{N}$ such that $B_{ij} = 0$ when $|i - j| > r$. Thus, the only nonzero entries of B are the entries B_{ij} such that $|i - j| \leq r$. Interpreting the indices i and j as vertices on the claw graph C^r_{2n}, it is clear that these nonzero entries of B in some sense correspond to connected vertices of C^r_{2n}. This connection can be made explicit by defining the weight of an edge in a claw graph.

Definition 6.8. *Let* $C^r_{2n} = (V, E)$ *be the* r*-claw on* $2n$ *vertices and let* B_{ij} *denote an entry of a banded matrix. The* edge weight *of* $ij \in E$ *is defined as* B_{ij}*.*

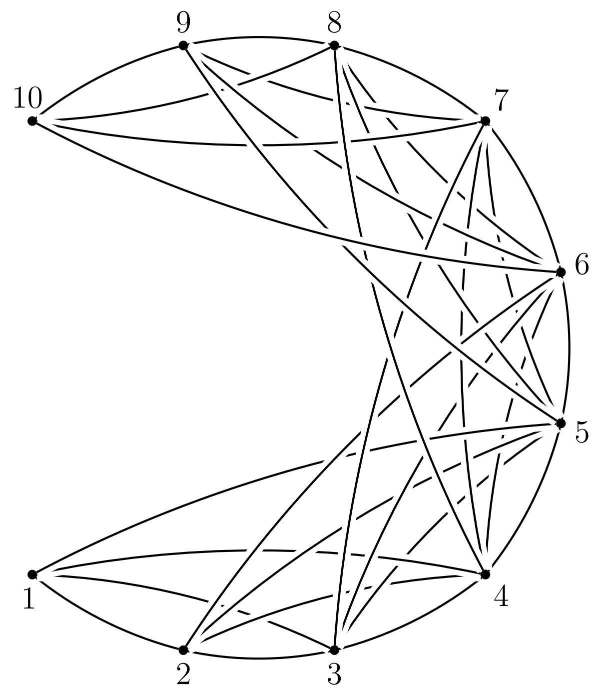

Figure 1: the 4-claw graph on 10 vertices

This simple idea can be used to determine Pfaffian image sequences in a direct fashion. As an alternative to characterizing image sequences by recurrence relations, as was done in Section 6, we can directly "count" the perfect matchings and their corresponding weights to arrive at closed-form solutions for the same image sequences.

Proposition 6.9. *Let a_n form the sequence satisfying the 3-term linear recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ with $Pf(a_1, a_2, a_3, ...) = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ...)$. Recall the matchings polynomial of a graph G given by $\mu(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k p(G, k)(x)^{n-2k}$, where $p(G, k)$ is the number of k-matchings in G. Then the n^{th} term in the Pfaffian image sequence $\tilde{a}_n$ is simply the matchings polynomial of a path on n vertices evaluated at 2:*

$$\tilde{a}_n = \mu(P_n, 2) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} 2^{n-2k}$$

Proof. By Proposition 4.4, it follows that $Pf(a_1, a_2, a_3, ...) = Pf(2, 1, 0, 0, ...)$. Consider the upper triangular of the $2n \times 2n$ matrix B corresponding to $Pf(2, 1, 0, 0, ...)$:

$$B = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 & \cdots & 0 \\ * & 0 & a_{23} & a_{24} & \cdots & 0 \\ * & * & 0 & a_{34} & \cdots & 0 \\ * & * & * & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 & 0 & \cdots & 0 \\ * & 0 & 2 & 1 & \cdots & 0 \\ * & * & 0 & 2 & \cdots & 0 \\ * & * & * & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & 0 \end{pmatrix} \tag{20}$$

The only edges on the complete graph K_{2n} that exist according to Eqn 2. are the those of the form $(i, i+1)$ and of the form $(i, i+2)$, so we are in fact considering the

2-claw graph C_{2n}^2. This places a constraint on the types of edges on K_{2n} that make a contribution to the particular Pfaffian under consideration.

These entries come under two classes. For edges corresponding to entries along the first diagonal of B, only those of the form $(2i-1, 2i)$ can be picked (to see why, consider the implication of picking an edge of the form $(2i, 2i+1)$). Denote these n edges as $\{v_i\}$, defining the weight of each v_i as 2. For edges corresponding to entries along the second diagonal of B, only edge pairs of the form $(2i-1, 2i+1), (2i, 2i+2)$ can be picked (to see why, again consider the implication of any other choice). Denote these $n-1$ edges as $\{e_i\}$, defining the weight of each e_i as -1.

Define a path $P_n = (V, E)$ on n vertices with a vertex set $V = \{v_i\}$, and with a edge set $E = \{e_i\}$. Choosing a perfect matching on C_{2n}^2 that contains k edge pairs of the form $(2i-1, 2i+1), (2i, 2i+2)$ is thus equivalent to choosing a k-matching on P_n. Noting that the vertex weights and edge weights on P_n are 2 and -1 respectively, it is then easy to see that the Pfaffian under consideration is equivalent to the matchings polynomial of P_n evaluated at 2.

To complete the proof, it remains to compute $p(P_n, k)$, the number of k-matchings on P_n. Consider a k-matching in P_n. Contract each edge in this k-matching, giving rise to $n-k$ vertices on an edge-less graph, each of which may or may not correspond to one of the k matched vertices in P_n. Thus, $p(P_n, k) = \binom{n-k}{k}$. The result follows. ☐

Note that with the foregoing proof, we reduced the problem of considering perfect matchings on a claw graph to that of considering matchings on a path. Though it has been shown for this specific case that the Pfaffian is merely the matchings polynomial of a certain graph evaluated at some real number, it is not clear that this pattern continues for more complicated cases.

Proposition 6.9 can be further generalized by considering a sequence satisfying a general 3-term linear recurrence relation. By building on the result of Proposition 6.9, a closed-form solution for the Pfaffian image sequence of this more general input sequence can also be found.

Proposition 6.10. *Let a_n form the 3-term recurrence sequence satisfying the recurrence relation $a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \alpha_3 a_{n-3}$ then:*

$$\tilde{a}_n = \alpha_3^n \mu\left(P_n, \frac{1+\alpha_2}{\alpha_3}\right) = \alpha_3^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \left(\frac{1+\alpha_2}{\alpha_3}\right)^{n-2k}$$

Proof. By Proposition 6.9, it is known that for the special case of $\alpha_2 = \alpha_3 = 1$,:

$$\tilde{a}_n = \mu\left(P_n, 1+\alpha_2\right) \tag{21}$$

where the matchings polynomial of a graph G is given by $\mu(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k p(G, k)(x)^{n-2k}$.

Note that the banded matrix B created by the reduction algorithm described in Proposition 4.4 is of the form:

$$B = \begin{pmatrix} 0 & 1+\alpha_2 & \alpha_3 & \cdots \\ -(1+\alpha_2) & 0 & 1+\alpha_2 & \cdots \\ -\alpha_3 & -(1+\alpha_2) & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{22}$$

We can normalize this n $\times$ n matrix by dividing by α_3. By Proposition 3.5, this leads to an α_3^n multiplicative factor in the expression for $\tilde{a}_n$. This normalization also results in a new banded matrix B'.

$$B' = \begin{pmatrix} 0 & \frac{1+\alpha_2}{\alpha_3} & 1 & \cdots \\ \frac{1+\alpha_2}{\alpha_3} & 0 & \frac{1+\alpha_2}{\alpha_3} & \cdots \\ -1 & -\frac{1+\alpha_2}{\alpha_3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{23}$$

Note that the values on the first diagonal are now $\frac{1+\alpha_2}{\alpha_3}$ instead of $1+\alpha_2$. Given (21), we can replace the $1+\alpha_2$ by $\frac{1+\alpha_2}{\alpha_3}$ to arrive at

$$\tilde{a}_n = \alpha_3^n \mu \left(P_n, \frac{1+\alpha_2}{\alpha_3} \right)$$

□

We can, in principle, determine closed-form solutions for an image sequence corresponding to the r-claw graph for any $r \in \mathbb{N}$. However, the counting process described in Proposition 6.9 becomes increasingly more difficult as r increases (try computing a closed-form solution for an image sequence corresponding to the 4-claw!). However, for the specific 2-claw case, we note that the matchings polynomial of a path is in fact an object known as the Chebyshev polynomial. In particular, the polynomial cited in Proposition 6.9 is the Chebyshev polynomial of the second kind $U_n(x)$. This is significant in that this polynomial has been extensively studied, and so we have the following definition in terms of a generating function and corresponding recurrence relation.

$$\sum_{k=0}^{\infty} U_n(t) = \frac{1}{1-2xt-t^2} \tag{24}$$

$$\begin{aligned} &U_0(t) = 1 \\ &U_1(t) = 2t \\ &U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t) \end{aligned} \tag{25}$$

Even though we may not easily be able to explicitly determine the Pfaffian polynomials as in Proposition 6.9, it turns out that we will be able to determine the generating function, and thus the recurrence relation satisfied by these polynomials. With the appropriate concepts in place, we can make a definite statement about the recurrence relations satisfied by the image sequences of any sequence satisfying a linear recurrence relation with constant coefficients.

The first of these concepts is that of a state defined on a claw graph. These states can be expressed combinatorially in terms of a binary representation.

Definition 6.11. *For any positive integer k, the* binary representation *of k is the sequence $a_m a_{m-1} \ldots a_0$ of 0s and 1s such that $a_m = 1$ and $k = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2 + a_0$. The k^{th}* state *is defined as the binary representation of the integer k.*

Example 6.12. *Consider the integer $k = 5$; since $5 = 2^2 + 2^0$, we say the binary representation of 5 has ones in positions 0 and 2.*

The state graph on a claw graph can now be defined. Recall that the claw graph C^r_{2n} is a subgraph of the complete graph K_{2n}. The state graph, in turn, is defined as a subgraph of C^r_{2n}.

Definition 6.13. *Fix $r \geq 1$ and k with $0 \leq k \leq 2^{r-1} - 1$, and let b denote the number of ones in the binary representation of k. The k^{th}* state graph of the r-claw on $2n$ vertices, *$S^{r,k}_{2n}$, is the graph formed from the r-claw on $2n + b$ vertices by removing vertex i (and all of its incident edges) whenever the binary representation of k has a 1 in position $i - 2$.*

Example 6.14. *Let $n = 3$, $k = 1$, and $r = 2$. By the binary representation of 1 has a single one, so $b = 1$, and it is in position 0. Therefore, to construct the 1^{st} state graph of the 2-claw on 6 vertices, we begin with the 2-claw on $2n + b = 7$ vertices and remove the vertex numbered $0 + 2 = 2$.*

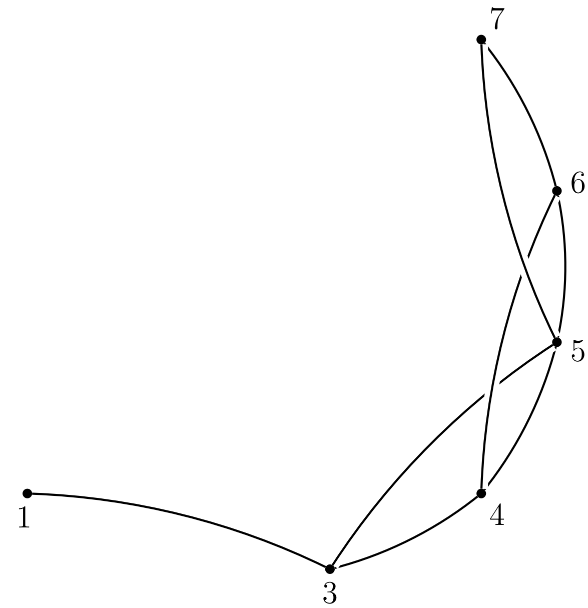

Figure 2: the 1^{st} state graph of the 2-claw on 6 vertices

The usefulness of the state concept derives in part from the fact that there are only a finite number of k states in an r-claw for any given $r \in I\!N$. In particular, by Definition 6.13, there are 2^{r-1} possible k states associated with an r-claw graph, labeled $k = 0, 1, ..., 2^{r-1} - 1$. Representations of the k^{th} state for $0 \leq k \leq 15$ is given in the following table, where a state graph's deleted vertices are denoted by filled dots.

	1	2	3	4	5	6	...	$2n$
0	○	○	○	○	○	○	...	○
1	○	●	○	○	○	○	...	○
2	○	○	●	○	○	○	...	○
3	○	●	●	○	○	○	...	○
4	○	○	○	●	○	○	...	○
5	○	●	○	●	○	○	...	○
6	○	○	●	●	○	○	...	○
7	○	●	●	●	○	○	...	○
8	○	○	○	○	●	○	...	○
9	○	●	○	○	●	○	...	○
10	○	○	●	○	●	○	...	○
11	○	●	●	○	●	○	...	○
12	○	○	○	●	●	○	...	○
13	○	●	○	●	●	○	...	○
14	○	○	●	●	●	○	...	○
15	○	●	●	●	●	○	...	○

Note that the 2nd column corresponds to the 2^0 factor of the binary expansion of k. Similarly, the 3rd corresponds to the 2^1 factor and generally the i^{th} column corresponds to the 2^{i-2} for $2 \leq i \leq 2n$.

In order to explicitly see the connection between states graphs and claw graphs, consider constructing perfect matchings on the r-claw graph. We begin with $2n$ ordered vertices and we add edges. Let's adhere to the convention that each edge added must be connected to the unused vertex with the least index, denoted v_i. Then because each edge we consider adding can have at maximum length r, we need only consider the vertices $v_i, v_{i+1}, \ldots, v_{i+r}$. Clearly there are only a finite number of states (or series of used and unused vertices) for these vertices, and they are represented by all the possible binary expansions. Thus we can conceptualize the state graphs as subgraphs of the r-claw that represent the vertices and edges available to complete a perfect matching on the r-claw.

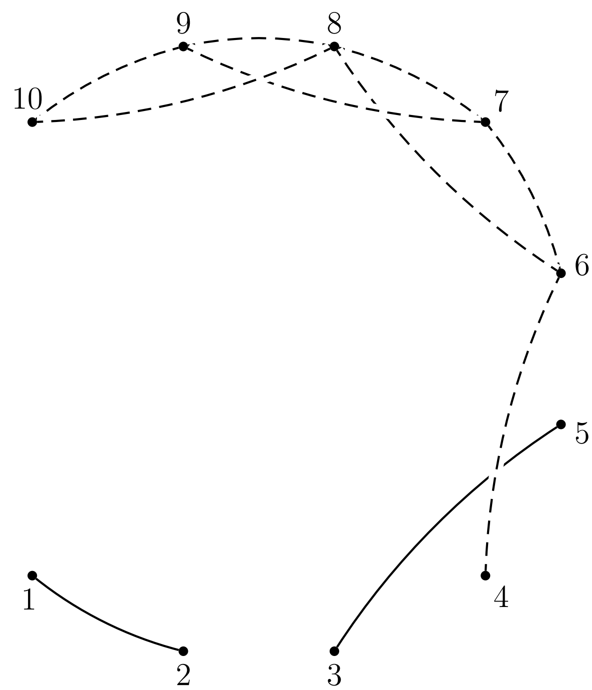

Figure 3: a partial perfect matching of C_{2n}^2 (bold); the 1^{st} state graph of the 2-claw on the remaining 6 vertices (dotted) is explicitly included to highlight that it represents possible completions of the perfect matching

Again, consider how we construct perfect matchings on the r-claw graph. Assume we have arrived at a partial perfect matching such as the one depicted in Figure 3. As in Figure 3, one can interpret the remaining vertices as a i^{th} state graph. Now, consider removing one edge from the partial perfect matching. We now have 2 more vertices added to the previous set of remaining vertices, and in fact, we can interpret these vertices as corresponding to a new j^{th} state graph.

What is essentially being described here is a recursive process by which states are expressed in terms of other states, which in turn may be expressible in terms of the original state. That is, for some state i, we can add an edge from the underlying claw graph to form another state j. This recursion idea can be used to motivate the concept of a state directed graph.

Definition 6.15. *The* state digraph of the r-claw $D^r = (V, E)$ *is the graph such that* $V = \{0, ..., 2^{r-1} - 1\}$,
$E = \{(i, j) | j^{th}$ *state graph can be constructed from* i^{th} *state graph by adding one edge*$\}$, *and* $w(i, j)$ *is equal to the signed weight of the edge in the claw graph used in such a construction of state* j *from state* i.

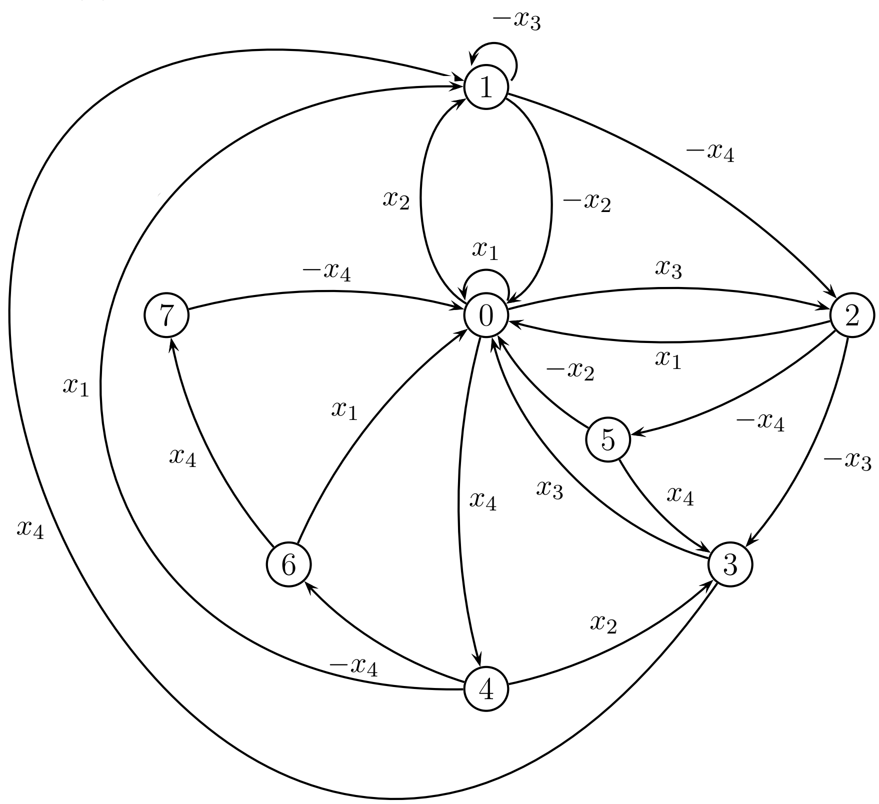

Figure 4: state digraph corresponding to the 4-claw graph

Before the full potential of the state directed graph can be realized, we must first find some way of describing the edge in the claw graph used to construct a state j from another state i. Though the weight is straightforward to determine, the sign is less so. We will take the sign of a perfect matching α to be the same as the sign of its corresponding Pfaffian permutation π. The sign of π is defined in the usual fashion as $sgn(\pi) = (-1)^{i(\pi)}$, where $i(\pi)$ is the number of inversions in π.

Considering the pictorial nature of graphs, it is useful to relate the quantity $sgn(\alpha)$ to a pictorial property of the graph itself. It turns out that such a property exists.

Definition 6.16. *Let* K_{2n} *be the complete graph on* $2n$ *vertices. For* $i < j$ *and* $k < l$, *define two edges* ij *and* kl *in* K_{2n} *as* crossed *whenever* $i < k < j < l$ *or* $k < i < l < j$. *The* crossing number of a perfect matching α in K_{2n} is the number of crossed edges in α, denoted $cr(\alpha)$.

Note that given the specific drawing of claw graphs where each vertex is placed on the circumference of a circle, and each edge is drawn at the interior of this circle, the pictorial nature of a crossing is immediately clear (see Figure 5). The sign of a perfect matching and its crossing number can now be explicitly related.

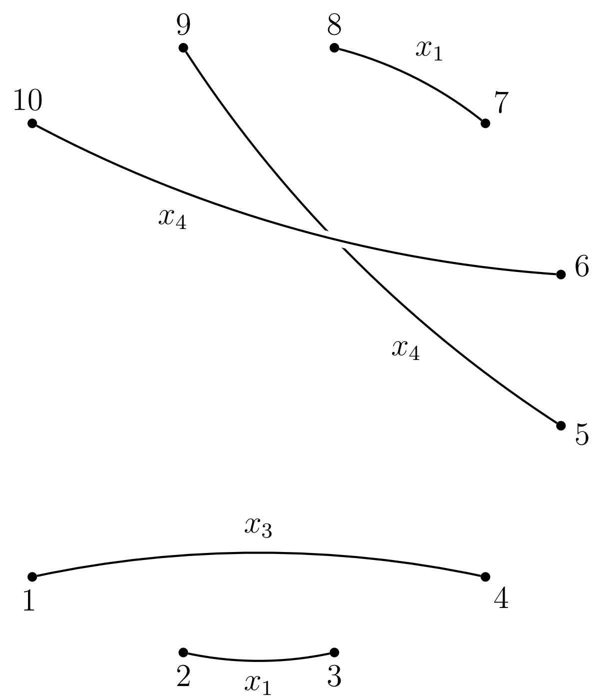

Figure 5: perfect matching in the 4-claw graph with one crossing

Lemma 6.17. *Let K_{2n} be the complete graph on 2n vertices, and let $cr(\alpha)$ be the crossing number of the perfect matching α on K_{2n}; then the sign of α can be expressed in terms of the crossing number of α*

$$sgn(\alpha) = (-1)^{cr(\alpha)}$$

Proof. Let $\alpha = (u_1, v_1)(u_2, v_2)...(u_n, v_n)$ be a perfect matching of K_{2n}, corresponding to a Pfaffian permutation π such that $\pi = u_1v_1u_2v_2...u_nv_n$.

Let $i(\pi)$ be the number of inversions in π. In addition to the sign of a perfect matching $sgn(\alpha)$ and the sign of a permutation $sgn(\pi)$, we will call the sign of a nonzero integer $sgn(q)$, defined as $sgn(q) = 1$ if $q > 0$ and $sgn(q) = -1$ if $q < 0$. Then

$$sgn(\alpha) = sgn(\pi) = (-1)^{i(\pi)} = \prod_{1 \le i \le j \le 2n} sgn(\pi(j) - \pi(i)). \tag{26}$$

This expression can be further decomposed by considering all possible edges of α, giving

$$sgn(\alpha) = \prod_{1 \le i \le j \le n} sgn(u_j - u_i) sgn(v_j - u_i) sgn(u_j - v_i) sgn(v_j - v_i) \times \prod_{1 \le i \le n} sgn(v_i - u_i). \tag{27}$$

By Definition 6.3 of the Pfaffian permutation, the first vertex label in a pair (u_i, v_i) is always the smaller, such that $u_i < v_i$ for $\forall i$. This implies that

$$sgn(\alpha) = \prod_{1 \le i \le j \le n} sgn(u_j - u_i) sgn(v_j - u_i) sgn(u_j - v_i) sgn(v_j - v_i) \tag{28}$$

Consider a crossing in a fixed drawing of α, constructed by placing the vertices of K_{2n} at distinct points of a circle, with edges drawn at the interior of the circle. Given such a drawing, Definition 6.16 dictates that the edges (u_i, v_i) and (u_j, v_j) of α cross iff

$$u_i < u_j < v_i < v_j \;\text{ or }\; u_j < u_i < v_j < v_i \tag{29}$$

Comparing Equation (29) with Definition 6.16, we find that the edges (u_i, v_i) and (u_j, v_j) cross iff

$$sgn(u_j - u_i) sgn(v_j - u_i) sgn(u_j - v_i) sgn(v_j - v_i) = -1. \tag{30}$$

It follows that $sgn(\alpha) = (-1)^{cr(\alpha)}$. □

Given the foregoing definitions, the states of a claw graph can now be described in terms of a polynomial in $x_1, x_2, ..., x_n$, the entries of the reduced matrix B as described in Proposition 4.3. This will prove useful, since the Pfaffian is also a polynomial in $x_1, x_2, ..., x_n$.

Definition 6.18. *Fix $n \ge 1$, $k \ge 0$, and r. Then the* state polynomial $f^k_{2n}(x_1, \ldots, x_r)$ *is defined by*

$$f^k_{2n}(x_1, \ldots, x_r) = \sum_{\pi} sgn(\pi) \prod_{ij \in \pi} x_{|i-j|}, \tag{31}$$

where the sum is over all perfect matchings π of the kth state graph of the r-claw on $2n$ vertices.

Note that the state polynomial is a generalization of the Pfaffian polynomial; in particular, $f^0_{2n}(x_1, \ldots, x_r)$ is the n^{th} term in the Pfaffian image sequence of the sequence $(x_1, x_2, \ldots, x_r, 0, 0, 0, \ldots)$. The construction of state polynomials is demonstrated by the following example.

Example 6.19. *By referring to Figure 4 and using Lemma 6.17, we can construct the state polynomials of the 4-claw in terms of other state polynomials. Note that parameters are suppressed (eg: $f^0_{2n} = f^0_{2n}(x_1, x_2, x_3, x_4)$).*

$$\begin{aligned}
f^0_{2n} &= x_1 f^0_{2n-2} + x_2 f^1_{2n-2} + x_3 f^2_{2n-2} + x_4 f^4_{2n-2} \\
f^1_{2n} &= -x_2 f^0_{2n-2} - x_3 f^1_{2n-2} - x_4 f^2_{2n-2} \\
f^2_{2n} &= x_1 f^0_{2n-2} - x_3 f^3_{2n-2} - x_4 f^5_{2n-2} \\
f^3_{2n} &= x_3 f^0_{2n-2} + x_4 f^1_{2n-2} \\
f^4_{2n} &= x_1 f^1_{2n-2} + x_2 f^3_{2n-2} + x_4 f^6_{2n-2} \\
f^5_{2n} &= -x_2 f^0_{2n-2} + x_4 f^3_{2n-2} \\
f^6_{2n} &= x_1 f^0_{2n-2} + x_4 f^7_{2n-2} \\
f^7_{2n} &= -x_4 f^0_{2n-2}
\end{aligned}$$

The previous example suggests that one can construct vectors of state polynomials such that the state vector on $2n$ vertices is produced by the action of some matrix on the state vector on $2n-2$ vertices. This matrix is, in fact, the adjacency matrix of the state directed graph.

Definition 6.20. *The* adjacency matrix *A of a directed graph D is given by $A_{ij} = w(i, j)$ for $i, j \in V(D)$.*

Consider the adjacency matrix A of the state digraph; one can easily verify that the state vectors satisfy a linear dynamical equation of the following form.

$$\begin{pmatrix} f_{2n}^1(x_1, ...) \\ f_{2n}^2(x_1, ...) \\ \vdots \\ f_{2n}^{2^{r-1}}(x_1, ...) \end{pmatrix} = A \begin{pmatrix} f_{2n-2}^1(x_1, ...) \\ f_{2n-2}^2(x_1, ...) \\ \vdots \\ f_{2n-2}^{2^{r-1}}(x_1, ...) \end{pmatrix} = A^n \begin{pmatrix} f_0^1(x_1, ...) \\ f_0^2(x_1, ...) \\ \vdots \\ f_0^{2^{r-1}}(x_1, ...) \end{pmatrix} \tag{32}$$

With these definitions in place, we can begin to consider the main result of this paper, the Pfaffian Recurrence Theorem.

7 The Pfaffian Recurrence Theorem

In this section, we prove a theorem concerning the Pfaffian image of a general class of sequences, namely those that satisfy linear homogeneous recurrence relations with constant coefficients. Before proving this theorem, a few definitions are in order.

Definition 7.1. *A* normalized rational function *is a rational function $\frac{p(x)}{q(x)}$ such that $p(x)$ and $q(x)$ have no common factors and $q(0) = 1$.*

Note that any rational function can be normalized in this way.

Definition 7.2. *The* generating function *of a sequence a_n is the power series $F(x) = \sum_{n=0}^{\infty} a_n x^n$.*

The relation between generating function and the sequences that we are studying, namely those that satisfy linear recurrence relations, becomes apparent with the following lemma.

Lemma 7.3. *Let $F(x)$ be the generating function of the sequence a_n, and let $p(x)$ be an M^{th} order polynomial $p(x) = p_0 + p_1 x + ... + p_M x^M$, and let $q(x)$ be an N^{th} order polynomial $q(x) = 1 - (q_1 x + q_2 x + ... + q_N x^N)$. Then $F(x) = \frac{p(x)}{q(x)}$ is a normalized rational function, iff a_n satisfies the N-term homogeneous linear recurrence relation with constant coefficients given by*

$$a_n - q_1 a_{n-1} - ... - q_N a_{n-N} = 0.$$

Proof. (⇒) Let $F(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{p(x)}{q(x)}$, where

$$p(x) = p_0 + p_1 x + ... + p_M x^M \tag{33}$$

$$q(x) = 1 - (q_1 x + q_2 x + ... + q_N x^N) \tag{34}$$

Then we have

$$(a_0 + a_1 x + a_2 x^2 + ...)(1 - q_1 x - q_2 x - ... - q_N x^N) = p_0 + p_1 x + ...p_M x^M. \tag{35}$$

Comparing coefficients with the same powers of x, one finds

$$\begin{aligned} p_0 &= a_0 \\ p_1 &= a_1 - q_1 a_0 \\ p_2 &= a_2 - a_1 q_1 - a_0 q_2 \\ &\vdots \\ p_M &= a_M - ... - a_0 q_M \end{aligned}$$

Note that for $n > M$, we have $p_n = 0$. Thus, for $n > M$,

$$a_n - q_1 a_{n-1} - ... - q_N a_{n-N} = 0. \tag{36}$$

(⇐) Assume $a_n - q_1 a_{n-1} - ... - q_N a_{n-N} = 0$ for some coefficients q_n. Define p_m such that

$$\begin{aligned} p_0 &= a_0 \\ p_1 &= a_1 - q_1 a_0 \\ p_2 &= a_2 - a_1 q_1 - a_0 q_2 \\ &\vdots \\ p_M &= a_M - ... - a_0 q_M \end{aligned}$$

Define $F(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$, $p(x)$ according to (33), and $q(x)$ according to (34). It then follows that

$$\begin{aligned} F(x)q(x) &= (a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...)\left(1 - (q_1 x + q_2 x + ... + q_N x^N)\right) \\ &= a_0 + (a_1 - q_1 a_0)x + (a_2 - a_1 q_1 - a_0 q_2)x^2 + ... + (a_M - ... - a_0 q_M)x^M \\ &= p_0 + p_1 x + p_2 x^2 + ... + p_M x^M \\ &= p(x) \end{aligned}$$

Thus, $F(x)$ is a normalized rational function, given by

$$F(x) = \frac{p(x)}{q(x)}. \tag{37}$$

□

In addition to having a direct correspondence with recurrence relations, the generating function for a sequence of matrix entries can in fact be written in closed form (2).

Lemma 7.4. *Let $F_{ij}(t) = \sum_{n=0}^{\infty}(A^n)_{ij}t^n$ be the generating function defined in terms of the ij^{th} entry of the n^{th} power of the matrix A, and let $(A : j, i)$ denote the classical adjoint of A, obtained by removing the j^{th} row and the i^{th} column of A. Then*

$$F_{ij}(t) = \frac{(-1)^{i+j}\det\left((I - tA) : j, i)\right)}{\det\left(I - tA\right)}$$

Proof. Note that $F_{ij}(t)$ is the ij^{th} entry of the matrix $\sum_{n=0}^{\infty}(A^n)t^n$. Evaluating the geometric series yields the form $\sum_{n=0}^{\infty}(A^n)t^n = (I - tA)^{-1}$, where I is the identity matrix. As a corollary of Cramer's Rule (2), it is well known that for any matrix A, we have $A^{-1} = \frac{\det(A:j,i)}{\det(A)}$. Applying this to the expression $F_{ij}(t) = (I - tA)^{-1}$ yields the result. □

The primary import of Lemma 7.4 is that the generating function $F_{ij}(t)$ is a rational function (note that both the numerator and denominator of $F_{ij}(t)$ are determinants, which are polynomials by definition). This, coupled with Lemma 7.3, will be key in proving the main theorem, which can now be stated.

Theorem 7.5. *Let (a_n) be a sequence satisfying a homogeneous linear recurrence relation with constant coefficients; then $Pf\left((a_n)\right)$ also satisfies a homogeneous linear recurrence relation with constant coefficients.*

Proof. By Proposition 4.3, the matrix A_n, such that $Pf((a_n)) = Pf(A_n) = \tilde{a}_n$ where a_n satisfies an k-term recurrence relation, can be reduced to a banded skew-symmetric Toeplitz matrix B_n with $r \leq k$ nonzero homogenous diagonals except at the bottom-right-most corner of size at most $M \times M$, where $M \leq r$. Note that the r-claw graph corresponding to this banded matrix B that has $W = 2^{r-1}$ states.

Thus, there is a finite number of state polynomials, denoted by

$$\begin{gathered} f_{2n}^{0}(x_1, ...) \\ f_{2n}^{1}(x_1, ...) \\ \vdots \\ f_{2n}^{W-1}(x_1, ...) \end{gathered}$$

It is important to reiterate here that the sequence of 0^{th} state polynomials corresponds to the Pfaffian image sequence $f_{2n}^{0}(x_1, ...) = \tilde{a}_n$.

Since there are only a finite number of state vectors affected by the $M \times M$ bottom-right corner, namely $2M$ of them, then iterating the first equality in Equation (32) gives a matrix equation for the state vector on $2n$ vertices, in terms of the state vector on $2M$ vertices with entries given by constants denoted $f_{2M}^1, ..., f_{2M}^{W-1}$.

$$\begin{pmatrix} f_{2n}^0(x_1, ...) \\ f_{2n}^1(x_1, ...) \\ \vdots \\ f_{2n}^{W-1}(x_1, ...) \end{pmatrix} = A^{n-M+1} \begin{pmatrix} f_{2M}^0 \\ f_{2M}^1 \\ \vdots \\ f_{2M}^{W-1} \end{pmatrix} \tag{38}$$

(38) implies that the sum $\tilde{a}_n = f_{2n}^0 = (A^{n-M+1})_{1,1} f_{2M}^0 + (A^{n-M+1})_{1,2} f_{2M}^1 + ... + (A^{n-M+1})_{1,W} f_{2M}^{W-1}$ contains all the perfect matchings that construct the 1-state on $2n$ vertices from the 1-state on 0 vertices (note that in the special case where there is no inhomogeneity in the bottom-right corner, (32) implies that $(A^n)_{11}$ is the entry that contains all the perfect matchings that construct the 1-state on $2n$ vertices from the original 1-state on 0 vertices).

Let $F_{ij}(t) = \sum_{n=0}^{\infty} (A^{n-M+1})_{ij} t^n = \sum_{n=0}^{\infty} \tilde{a}_n t^n$. By Lemma 7.4, each $F_{ij}(t)$ is a rational function (which can be normalized). Also note from this lemma that each $F_{ij}(t)$ has the same denominator $\det(I - tA)$. In particular, $\sum_{n=0}^{\infty} \tilde{a}_n t^n$ is also a rational generating function (which can be normalized), with denominator $\det(I - tA)$.

By Lemma 7.3, the sequence $(\tilde{a}_n)$ then satisfies a linear recurrence relation with constant coefficients for $n > M$. □

A further statement about the Pfaffian image can be made, regarding the length of the recurrence relation that this image sequence satisfies.

Corollary 7.6. *Let (a_n) be a sequence satisfying an r-term recurrence relation; then $Pf((a_n))$ satisfies a 2^{r-1}-term recurrence relation.*

Proof. By Definitions 6.15 and 6.20, the adjacency matrix A of the state directed graph is a $2^{r-1} \times 2^{r-1}$ matrix.

Consider the denominator of the generator $F_{11}(t)$.

Recall the definition of the determinant of an $n \times n$ matrix A in terms of the permutations $\pi \in Sym(\{1, 2, ..., n\})$, given by $\det(A) = \sum_{\pi} sgn(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}$

Since $(I - tA)$ is a $2^{r-1} \times 2^{r-1}$ matrix with entries of 1s on the main diagonal and with all other entries involving factors of t, it is clear that $\det(I - tA)$ is a polynomial of degree at most 2^{r-1}, of the form $q(x) = 1 + q_1 x + q_2 x + ... + q_{2^{r-1}} x^{2^{r-1}}$.

By Lemma 7.3, the output recurrence relation satisfies a 2^{r-1}-term recurrence relation. □

We have now a concise way of describing the action of the Pfaffian transformation on sequences satisfying linear recurrence relations with constant coefficients. In particular, Theorem 7.5 assures us that the image sequence $(\tilde{a}_n)$ resulting from such a transformation satisfies a linear recurrence relation with constant coefficients. However, this only gives us the general pattern satisfied by $(\tilde{a}_n)$ for sufficiently large n. In fact, the proof of Theorem 7.5 assumes that, in general, a finite number of initial terms of $(\tilde{a}_n)$ do not satisfy any particular recurrence relation. Since they are only finite in number, these terms can be computed using efficient computer algorithms.

8 Generating Perfect Matchings of K_{2n}

In this section we discuss how to generate perfect matchings on a complete graph with $2n$ vertices, whose vertices are ordered 1 to $2n$. We first consider how to generate these Pfaffian Permutations. To make things simple, we first decide to order our permutations lexicographically. Let π_i indicate the i^{th} Pfaffian Permutation of $1, 2, \ldots, 2n$ for $0 \leq i < (2n-1)!!$. Then our first permutation, π_0, is the identity permutation and our last,$\pi_{(2n-1)!!-1}$ is $(1\ 2n\ 2\ 2n-1\ \ldots\ n\ n+1)$.

Example 8.1. *The Pfaffian Permutations of* $1, 2, 3, 4, 5, 6$ *in order:*

i	π_i
0	(123456)
1	(123546)
2	(123645)
3	(132456)
4	(132546)
5	(132645)
6	(142356)
7	(142536)
8	(142635)
9	(152346)
10	(152436)
11	(152534)
12	(162345)
13	(162435)
14	(162534)

Consider constructing π_i one element at a time starting at the first position. Note that because of the definition of the Pfaffian Permutation $\pi_i(2i-1) < \pi_i(j)$ for $2i \leq j \leq 2n$. Then if we are constructing our permutation left to right each odd position is determined by the least unused value of $1, 2, \ldots, 2n$. We need only consider the values in the even positions.

Notice that there are $2n-1$ choices for position 2 and that each value occurs in $(2n-3)!!$ Pfaffian Permutations. Then we can partition the Pfaffian Permutations into sets $A_1, A_2, \ldots, A_{2n-1}$ where for each permutation,π, in set A_i, $\pi(2) = i+1$. Then note that $A_i = \{\pi_i | (i-1)(2n-3)!! < i \leq i(2n-3)!!\}$. This suggests that if we list the permutations of A_1, then A_2, and so on until A_{2n-1} we will list all the Pfaffian Permutations in lexicographic order.

For each A_i notice that we can partition these permutations based on position 4 into $2n-3$ sets of size $(2n-5)!!$ in the same way we partitioned the entire set based on position 2. Let A_{ij} be the j^{th} partitioning set of A_i. Then for a permutation π in A_{ij}, $\pi(4) = k$ where k is the j^{th} minimum value of the unused values.

We can continue creating these partitions for further even positions. In general for a set $A_{a_1 a_2 \ldots a_m}$ and a permutation π in that set, $\pi(2m) = k$ where k is the m^{th} minimum element of the unused elements of $1, 2, \ldots 2n$. Note that each π_i is an element of one set $A_{a_1 a_2 \ldots a_n}$. We can see that $a_1 = \lfloor \frac{i}{(2n-3)!!} \rfloor$. If we let $p = i \bmod (2n-3)!!$ we have determined the relative position of π_i within A_{a_1}, because, as we have noted, each A_j contains the sequential set of permutations $\pi_{(j-1)(2n-3)!!+1}, \ldots, \pi_{j(2n-3)!!}$. Then $a_2 = \lfloor \frac{p}{(2n-5)!!} \rfloor$. Note that we can repeat these steps to find the values of the remaining a_k.

These observations lead us to an algorithm for finding π_i explicitly:

1. Let $H = \{j | 1 \leq j \leq 2n\}$, $J = \{\}$, and c be an empty list. Also define $min_j(H)$ to be the j^{th} minimum value of H. That is, $min_j(H)$ is the minimum value of the remaining set after removing the minimum value of H j times. We see that $min(H) = min_0(H)$.

2. Set $pos = i$.

3. Let $a = min(H)$. Then remove a from H, and append a to c.

4. Let $l = (|H| - 2)!!$. Then set $d = \lfloor \frac{pos}{l} \rfloor$ and $pos = pos \bmod l$.

5. Let $b = min_d(H)$. Then remove b from H, and append b to c.

6. Repeat steps 3 through 5 until $|H| = 2$. Then append the two remaining elements of H in increasing order to c.

Then $\pi_i = c$.

We demonstrate this algorithm in the next example.

Example 8.2. *We generate the 23^{rd} Pfaffian Permutation of $1, 2, \ldots, 8$, $\pi_{23} = a_1 a_2 \ldots a_8$, as follows:*

1. *Let $H = \{1, 2, \ldots, 8\}$, $J = \{\}$. Set $pos = 23$. Remove $min(H) = 1$ from H and set $a_1 = 1$. Then $H = \{2, \ldots, 8\}$.*

2. *Set $d = \lfloor \frac{pos}{5!!} \rfloor = \lfloor \frac{23}{15} \rfloor = 1$. Then $pos = pos$ mod $5!! = 8$. Then $a_2 = min_1(H) = 3$. Remove 3 from H. Then $H = \{2, 4, 5, 6, 7, 8\}$.*

3. *Set $a_3 = min(H) = 2$. Remove 2 from H. Then $H = \{4, 5, 6, 7, 8\}$.*

4. *Set $d = \lfloor \frac{pos}{3!!} \rfloor = \lfloor \frac{8}{3} \rfloor = 2$. Then $pos = pos$ mod $3!! = 2$. Then $a_4 = min_2(H) = 6$. Remove 6 from H. Then $H = \{4, 5, 7, 8\}$.*

5. *Set $a_5 = min(H) = 4$. Remove 4 from H. Then $H = \{5, 7, 8\}$.*

6. *Set* $d = \lfloor \frac{pos}{1!!} \rfloor = \lfloor \frac{2}{1} \rfloor = 2$. *Then* $pos = pos \ mod\ 1!! = 0$. *Then* $a_6 = min_2(H) = 8$. *Remove* 8 *from* H. *Then* $H = \{5, 7\}$.

7. *Set* $a_7 = 5$ *and* $a_8 = 7$. *Then* $\pi_{23} = (13264857)$.

Clearly we can generate the entire list of Pfaffian Permutations on $1, 2, \ldots, 2n$ with this algorithm. This next method takes a different approach: given π_i it will generate π_{i+1}.

1. Let $k = 2n - 2$.

2. Let $H = \{a_j | j > k\}$. If $a_k = max(H)$ then decrease set $k = k - 2$ and repeat this step until $a_k \neq max(H)$ or $k = 0$. If $k = 0$, then $\pi_i = \pi_{(2n-1)!!-1}$ or the last Pfaffian Permutation. In this case, we will define π_{i+1} to be π_0, so stop and return identity permutation.

3. Let b the minimum value of H that is greater than a_k. Remove b from H and add a_k to H. Define a new permutation $\pi' = d_1 d_2 \ldots d_{2n}$ where $d_j = a_j$ for $1 \leq j < k$, $d_k = b$ and the remaining positions are the elements of H in increasing order.

Then $\pi_{i+1} = \pi'$.

Example 8.3. *Let* $\pi_i = (14283756)$. *Then to generate* $\pi_{i+1} = a_1 a_2 \ldots a_8$ *we follow the above algorithm. Notice that* $7 = max(\{7, 5, 6\})$ *and that* $8 = max(\{8, 3, 7, 5, 6\})$, *but* $4 \neq max(\{4, 2, 8, 3, 7, 5, 6\})$. *Then we let* $a_1 = 1$ *and set* $a_2 = 5$. *The rest of the permutation is ordered in increasing order. Then* $\pi_{i+1} = (15234678)$.

9 Code

In this discussion we will provide pseudo-code for some helpful algorithms we devised in order to better understand the Pfaffian. Specifically, we will have code examples for creating a skew symmetric matrix determined by some input sequence in our proper form, generating an entire list of Pfaffian Permutations for some permutation length, and finally we will demonstrate how to determine the Pfaffian of some input sequence with the graph theoretic method we described earlier.

1. Create the Pfaffian Matrix:

   procedure PFMAT(arg $a = (a_n)$) // input sequence of length $2n-1$

   create matrix$(2n \times 2n)$ A
   if $i < j$ **then**
   $\quad A_{i,j} \leftarrow a_{j-i}$
   else if $i > j$ **then**
   $\quad A_{i,j} \leftarrow -a_{j-i}$
   else
   $\quad A_{i,j} \leftarrow 0$

```
end if
return A
```

2. Generating the entire Pfaffian Permutation list of $1, 2, \ldots, 2n$ recursively:

```
procedure PERMLIST(arg templist1)          // list of 2n elements
if length templist = 2 then
    sort templist1
    return templist
else
    finishlist ← {}
    sort templist1
    for i = 1 to 2n − 1 do
        templist2 ← templist1
        tempperm ← ()
        tempvalue1 ← min templist2
        append tempvalue1 to tempperm
        remove tempvalue1 from templist2
        tempvalue2 ← i^th min of templist2
        append tempvalue2 to tempperm
        remove tempvalue2 from templist2
        append PERMLIST(templist2) to tempperm
        append tempperm to finishlist
    end for
    return finishlist
end if
```

3. Generating Perfect Matchings from Pfaffian Permutations:

```
procedure PFAFFIAN(arg a = (a_n))      //input sequence of length 2n−1
A ← PFMAT(a)
permlist ← PERMLIST({1, 2, ..., 2n − 2})
l ← length permlist
answer ← 0
for i = 1 to l do
    tempperm ← i^th element of permlist
    sign ← signature tempperm
    weight ← 1
    for j = 1 to n − 1 do
        x ← tempperm(2j − 1)
        y ← tempperm(2j)
        weight ← weight × A_{x,y}
    end for
    answer ← answer + sign × weight
end for
return answer
```

10 Directions for Future Work

We have developed mathematical machinery to describe the action of the Pfaffian transformation on the set of sequences satisfying linear recurrence relations with constant coefficients. This machinery can be used as a point of departure for further studies.

As a start, one can consider the action of the Pfaffian transformation on specific sequences. For example, there are some sequences that are invariant under the Pfaffian transformation, such as $(1,1,1,...) = Pf(1,1,1,...)$. It is interesting to think of other such sequences. In the same spirit, consider repeatedly applying the Pfaffian transformation to some sequence; the result of such a transformation is also unclear. As an example of a repeated application, note that $Pf(Pf(1,1,2,3,5,8,...)) = Pf(1,2,4,8,16,32,...) = Pf(1,1,1,1,1,1,...)$.

Instead of thinking of the Pfaffian as a map, we can shift gears and view it as a polynomial on the coefficients $x_1, x_2, ..., x_k$ in the input recurrence relation. In the discussion of states, we showed that this polynomial is in fact a special case of a more general set of polynomials which we called the state polynomials. Proposition 6.9 suggests another generalization. In particular, it is possible that the Pfaffian is a special case of the matchings polynomial of some graph, evaluated at some real number. In other words, if $Pf(a_n)$ is the Pfaffian of a sequence (a_n), then it's possible that there exists a graph G and $x \in \mathbb{R}$ such that $Pf(a_n) = \mu(G,x)$ where $\mu(G,x)$ is the matchings polynomial of some graph G evaluated at $x \in \mathbb{R}$.

As a final note, recall that the Pfaffian transformation Pf is a map from sequences to sequences. The Pfaffian Recurrence Theorem assures us that if we restrict Pf to the set of sequences S satisfying linear recurrence relations, we get a map to a set of sequences that also satisfy linear recurrence relations. However, we are not assured that the image of this map covers all of S. This is stated explicitly in the following conjecture.

Conjecture 10.1. *Let S be the set of all sequences satisfying linear recurrence relations with constant coefficients and let $Pf|_S : S \to S$ be the restriction of Pf to S. Then $Pf|_S$ is a surjective map.*

This conjecture becomes more credible when we consider the linear recurrence relations satisfied by the Pfaffian image sequences. The coefficients of these output recurrence relations are functions of the coefficients in the input recurrence relations satisfied by the input sequences. There seem to be, in some sense, enough "degrees of freedom" for $Pf|_S$ to be a surjective map, but again, this has yet to be proved. Further studies may resolve this question.

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