Boolean numbers and graph-induced sequences

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Winter–Spring 2014

In a series of recent papers, Tenner and Ragnarsson study finitely-generated Coxeter systems, an abstract version of reflection groups. To each such Coxeter system (a group-theoretic object), they associate a certain partial order (a set-theoretic object), to which they associate a simplicial complex (a topological object). This simplicial complex turns out to be simply a collection of spheres (of suitable dimension), and the number of those spheres (the "Boolean number") encodes important information about the original Coxeter system. To count these spheres, Tenner and Ragnarsson finally associate to each of these simplicial complexes a graph (a combinatorial object) and show that the Boolean number of a finitely-generated Coxeter system can be computed in a straightforward way using a recursive algorithm on the associated graph of that system.

Surprisingly, it turns out that some 'nice' families of graphs yield familiar sequences as their Boolean numbers. Tenner and Ragnarsson have shown that all of the following sequences can be realized as the Boolean numbers of certain families of graphs:

- the median Genocchi numbers g_n,
- the derangement numbers,
- the Fibonacci numbers F_n,
- the zero sequence,
- the Legendre-Stirling numbers $d(r, j)$, and
- the Stirling numbers of the second kind.

In this project, we will explore this rich network of connections among abstract algebra, geometry, topology, poset combinatorics, and graph theory. We'll begin by reading several of Tenner and Ragnarsson's papers (and relevant background material as appropriate), then dive into some computations and theoretical investigations. There are lots of possible questions to explore, including:

- For a specified sequence of combinatorially-interesting numbers (say, the Catalan numbers), can a 'nice' family of graphs be constructed whose Boolean numbers realize that sequence? (What does 'nice' mean here, anyway?)
- Under what conditions on the sequence can a family of such graphs be constructed?
- How does the Boolean number connect to other important graph-theoretic invariants like the connectivity number?
- Can the algorithm given by Tenner and Ragnarsson to compute the Boolean number of a graph be modified to compute other interesting graph invariants?

Although this project will intersect many areas of mathematics, it has no specific background prerequisites. There will be ample opportunities to move this project in the direction of your interests in any combination of graph theory, set combinatorics, algebra, or topology, and there will definitely be a role for some light programming as well.