FRACTAL SETS, HAUSDORFF MEASURE, AND APPLICATIONS

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Mandelbrot coined the word fractal from the Latin adjective *fractus*, or broken, and the term has since been used to describe sets that are irregular, exhibiting properties that are not well studied with classical geometry. Some well-known examples of fractals are the Cantor set and the von Koch snowflake curve. The Cantor set is constructed iteratively from a unit interval by first removing the open middle third of the interval and then successively removing the open middle third of each remaining subinterval at each step. The Cantor set has a number of interesting features, including that it is uncountable, yet has zero length. The von Koch curve can also be constructed iteratively from a unit interval, by replacing the middle third of each line segment with two sides of an equilateral triangle at each step, resulting in a set in the plane of infinite length and zero area. A key property of both of these examples is self-similarity; that is, each contains scaled copies of itself.

When studying fractals, it is important to have a notion of the sizes of sets, a way of measuring beyond the standard definitions of length, area, or volume. As the above examples show, traditional measures may fail to encapsulate the complex geometrical properties of fractal sets, and thus the notion of fractal dimension arises. As part of this project, we will consider different ways to define fractal dimensions and their associated measures, with an emphasis on the Hausdorff dimension. Besides developing and applying these concepts, we will investigate some related problems in mathematics today.

The only prerequisite for this project is real analysis, which can be taken concurrently. We will begin with some background in basic measure theory, as needed, and then consider how to define concepts of fractal measure and dimension. We will study Hausdorff measure in detail, and see how the Hausdorff dimension can reflect the complex properties of fractal sets. In particular, we will use the self-similarity properties of some fractals, such as the Cantor set and the von Koch curve, to explicitly compute their Hausdorff dimensions. In the case of the Cantor set, we will show that the Hausdorff dimension is $\log(2)/\log(3) \approx 0.63$, illustrating that this uncountable set is not quite zero-dimensional, but has zero length in dimension one. Similarly, for the infinite length von Koch curve we will show that the Hausdorff dimension is $\log(4)/\log(3) \approx 1.26$.

For other irregular sets without these self-similarity properties, the Hausdorff dimension can be very difficult to compute. An important example, with a broad range of applications, is that of Kakeya sets and the Kakeya conjecture. A Kakeya set is a subset of $\mathbb{R}^n$ containing a line segment in every direction. As part of this project, we will construct Kakeya sets that have zero area, and then consider their Hausdorff dimension. The Kakeya conjecture states that a Kakeya set in $\mathbb{R}^n$ must have Hausdorff dimension n. Roughly this says that despite having zero area, these sets are still "large." We will prove this assertion for $n = 2$, but the problem is still open for $n \geq 3$. Moreover, we will see how the existence of zero area Kakeya sets has led to some surprising results in analysis and how the Kakeya conjecture is connected to other problems in mathematics.