Restricted Symmetric Signed Permutations

Enumerations of Pattern-Avoiding Signed Permutations Invariant Under Certain Symmetry Subgroups

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Signed Permutations

Definition

A *permutation* of length n is an ordering of the numbers from 1 to n. For example, the permutations of length 3 are $123, 132, 213, 231, 312,$ and 321.

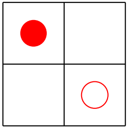
Definition

A *signed permutation* is a permutation where we can put bars over some of the entries. For example, the signed permutations of length 2 are $12, 1\bar{2}, \bar{1}2, \bar{1}\bar{2}, 21, 2\bar{1}, \bar{2}1,$ and $\bar{2}\bar{1}$.

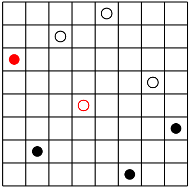
Pattern Avoidance

Definition

A signed permutation π *contains* another signed permutation (called a *pattern*) ρ if there is a substring of π with the same relative ordering and bar configuration as ρ. If π does not contain ρ, we say π *avoids* ρ.

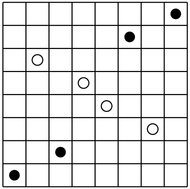


The pattern $\bar{2}1$

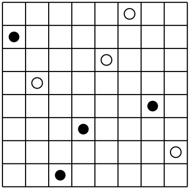


$62\bar{7}\bar{4}\bar{8}1\bar{5}3$ contains the pattern $2\bar{1}$

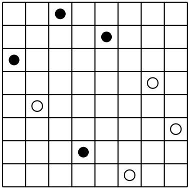
Symmetric Invariance



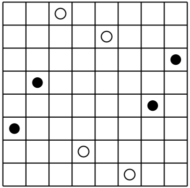
$1\bar{6}2\bar{5}\bar{4}7\bar{3}8$ is invariant under R_{180}.



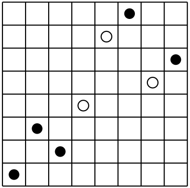
$7\bar{5}13\bar{6}\bar{8}4\bar{2}$ is invariant under $\overline{R_{180}}$.



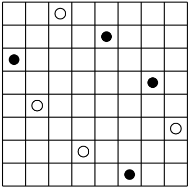
$6\bar{4}827\bar{1}\bar{5}\bar{3}$ is invariant under $\overline{D}$.



$35\bar{8}\bar{2}\bar{7}\bar{1}46$ is invariant under $\overline{R_{90}}$.



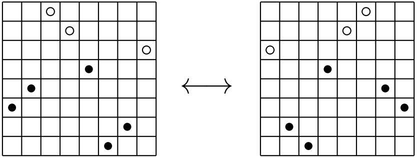
$132\bar{4}\bar{7}8\bar{5}6$ is invariant under D.



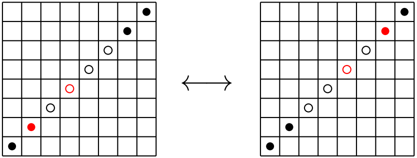
$6\bar{4}\bar{8}\bar{2}715\bar{3}$ is invariant under D and $\overline{D'}$.

Simplifications

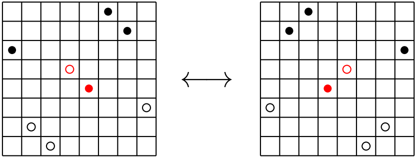
- Some symmetry operations, such as reflection over a vertical line, fix no permutations.



- If the permutation π avoids the pattern ρ and is invariant under the symmetry g, then π also avoids $g(\rho)$.



- If H is a symmetry subgroup, R is a set of patterns, and g is a symmetry in the normalizer of H, then for all $n \geq 0$, $|B_n^H(R)| = |B_n^H(g(R))|$.



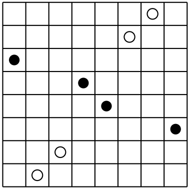
Permutations Invariant Under R_{180}

Result

$|B_{2k}^{180}(\bar{2}\bar{1}, 12)| =$
$|B_{2k+1}^{180}(\bar{2}\bar{1}, 12)| =$

Terms

1, 2, 2, 4, 6, 12, 20, 40, 70, 140



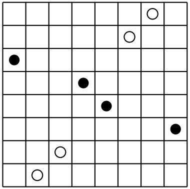
Permutations Invariant Under R_{180}

Result

$|B_{2k}^{180}(\bar{2}\bar{1}, 12)| = \binom{2k}{k}$.
$|B_{2k+1}^{180}(\bar{2}\bar{1}, 12)| = 2\binom{2k}{k}$.

Terms

1, 2, 2, 4, 6, 12, 20, 40, 70, 140



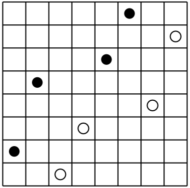
Permutations Invariant Under $\overline{R_{180}}$

Result

$|B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, 21)| =$

Even terms

1, 3, 10, 35, 126, 462, 1716, 6435



Permutations Invariant Under $\overline{R_{180}}$

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$|B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, 21)| = \binom{2k+1}{k}$.

Even terms

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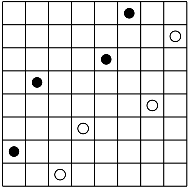


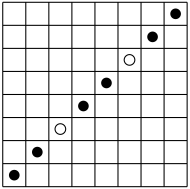
Permutations Invariant Under D and D'

Result

$|B_{2k}^{H}(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)| =$

Even terms

1, 2, 4, 8, 16, 32, 64, 128



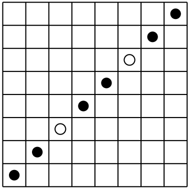
Permutations Invariant Under D and D'

Result

$|B^H_{2k}(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)| = 2^k.$

Even terms

1, 2, 4, 8, 16, 32, 64, 128



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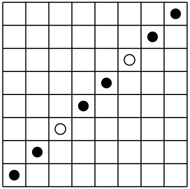
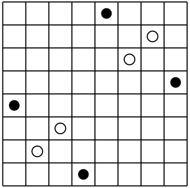


Result

$|B^H_{2k}(\bar{2}\bar{1})| =$

Even terms

1, 3, 11, 45, 201, 963, 4899, 26253



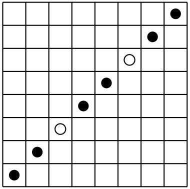
Permutations Invariant Under D and D'

Result

$|B^H_{2k}(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)| = 2^k.$

Even terms

1, 2, 4, 8, 16, 32, 64, 128

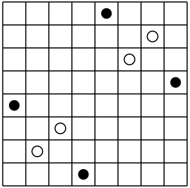


Result

$$|B^H_{2k}(\bar{2}\bar{1})| = 3|B^H_{2(k-1)}(\bar{2}\bar{1})| + 2(k-1)|B^H_{2(k-2)}(\bar{2}\bar{1})|.$$

Even terms

1, 3, 11, 45, 201, 963, 4899, 26253



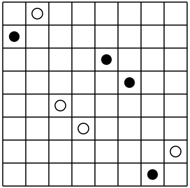
Permutations Invariant Under D and $\overline{D'}$

Result

$|B_{2k}^{W}(\bar{1}\bar{2}, 12)| =$

Even terms

1, 2, 4, 8, 16, 32, 64, 128



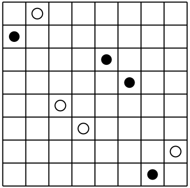
Permutations Invariant Under D and $\overline{D'}$

Result

$|B_{2k}^{W}(\bar{1}\bar{2}, 12)| = 2^k$.

Even terms

1, 2, 4, 8, 16, 32, 64, 128



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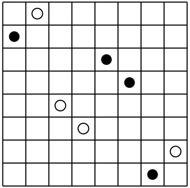
Even terms

1, 2, 4, 8, 16, 32, 64, 128

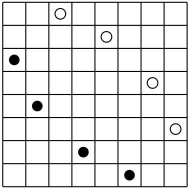


Result

$|B_{2k}^{W}(\bar{1}\bar{2}, \bar{1}2, 12)| =$

Even terms

1, 1, 2, 3, 6, 10, 20, 35



Permutations Invariant Under D and $\overline{D'}$

Result

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Even terms

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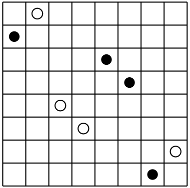
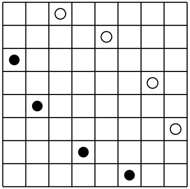


Result

$|B_{2k}^{W}(\bar{1}\bar{2}, \bar{1}2, 12)| = \binom{k}{\lfloor k/2 \rfloor}.$

Even terms

1, 1, 2, 3, 6, 10, 20, 35



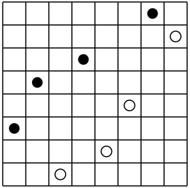
Permutations Invariant Under $\overline{D}$

Result

$|B_{2k}^{\overline{D}}(\bar{2}\bar{1}, \bar{2}1, 21)| =$

Even terms

1, 1, 2, 5, 14, 42, 132, 429



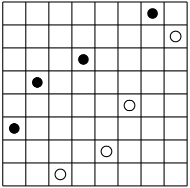
Permutations Invariant Under $\overline{D}$

Result

$|B_{2k}^{\overline{D}}(\bar{2}\bar{1}, \bar{2}1, 21)| = C_k.$

Even terms

1, 1, 2, 5, 14, 42, 132, 429



Open Questions

- Combinatorial proofs:
  - $|B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, 21)| = \binom{2k+1}{k}$
  - $|B_{2k}^{W}(\bar{1}\bar{2}, \bar{1}2, 12)| = \binom{k}{\lfloor k/2 \rfloor}$
- r-colored permutations for $r > 2$.
- Avoidances of length 2 and length 3

Acknowledgements

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