Restricted Symmetric Signed Permutations
Enumerations of Pattern-Avoiding Signed Permutations Invariant Under Certain Symmetry Subgroups

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Signed Permutations

Definition

A *permutation* of length $n$ is an ordering of the numbers from 1 to $n$. For example, the permutations of length 3 are 123, 132, 213, 231, 312, and 321.

Definition

A *signed permutation* is a permutation where we can put bars over some of the entries. For example, the signed permutations of length 2 are 12, 1\(\bar{2}\), \(\bar{1}2\), \(\bar{1}\bar{2}\), 21, 2\(\bar{1}\), \(\bar{2}1\), and \(\bar{2}\bar{1}\).  
Definition

A signed permutation $\pi$ contains another signed permutation (called a pattern) $\rho$ if there is a substring of $\pi$ with the same relative ordering and bar configuration as $\rho$. If $\pi$ does not contain $\rho$, we say $\pi$ avoids $\rho$.

The pattern $\overline{2}1$

62748153 contains the pattern $2\overline{1}$
Symmetric Invariance

16254738 is invariant under $R_{180}$.

75136842 is invariant under $R_{180}$.

64827153 is invariant under $D$.

35827146 is invariant under $R_{90}$.

13247856 is invariant under $D$.

64827153 is invariant under $D$ and $D^T$. 
Simplifications

- Some symmetry operations, such as reflection over a vertical line, fix no permutations.

- If the permutation $\pi$ avoids the pattern $\rho$ and is invariant under the symmetry $g$, then $\pi$ also avoids $g(\rho)$.

- If $H$ is a symmetry subgroup, $R$ is a set of patterns, and $g$ is a symmetry in the normalizer of $H$, then for all $n \geq 0$, $|B_n^H(R)| = |B_n^H(g(R))|$. 
Permutations Invariant Under $R_{180}$

**Result**

\[
|B_{2k}^{180}(21, 12)| = 2^{2k} \binom{2k}{k} \\
|B_{2k+1}^{180}(21, 12)| = 2^{2k+1} \binom{2k+1}{k+1}
\]

**Terms**

1, 2, 2, 4, 6, 12, 20, 40, 70, 140
Permutations Invariant Under $R_{180}$

Result

\[
|B_{2k}^{180}(\bar{21}, 12)| = \binom{2k}{k}.
\]

\[
|B_{2k+1}^{180}(\bar{21}, 12)| = 2\binom{2k}{k}.
\]

Terms

1, 2, 2, 4, 6, 12, 20, 40, 70, 140
Permutations Invariant Under $R_{180}$

Result

$$|B_{2k}^{180}(\bar{21}, \bar{21}, 21)| =$$

Even terms

1, 3, 10, 35, 126, 462, 1716, 6435
Permutations Invariant Under $R_{180}$

Result

$$|B_{2k}^{180}((21, 21, 21))| = \binom{2k+1}{k}.$$ 

Even terms

1, 3, 10, 35, 126, 462, 1716, 6435
Permutations Invariant Under $D$ and $D'$

**Result**

$$|B_{2k}^H(\bar{21}, \bar{21}, 2\bar{1}, 21)| =$$

**Even terms**

1, 2, 4, 8, 16, 32, 64, 128
Permutations Invariant Under $D$ and $D'$

Result

$$|B_{2k}^H(\overline{21}, \overline{21}, 2\overline{1}, 21)| = 2^k.$$ 

Even terms

1, 2, 4, 8, 16, 32, 64, 128
Permutations Invariant Under $D$ and $D'$

Result
\[ |B_{2k}^H(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)| = 2^k. \]

Even terms
1, 2, 4, 8, 16, 32, 64, 128

Result
\[ |B_{2k}^H(\bar{2}\bar{1})| = \]

Even terms
1, 3, 11, 45, 201, 963, 4899, 26253
Permutations Invariant Under $D$ and $D'$

Result

\[ |B_{2k}^H(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)| = 2^k. \]

Even terms

1, 2, 4, 8, 16, 32, 64, 128

Result

\[ |B_{2k}^H(\bar{2}\bar{1})| = 3|B_{2(k-1)}^H(\bar{2}\bar{1})| + 2(k - 1)|B_{2(k-2)}^H(\bar{2}\bar{1})|. \]

Even terms

1, 3, 11, 45, 201, 963, 4899, 26253
Permutations Invariant Under $D$ and $\overline{D'}$

Result

$$|B_{2k}^W(\overline{12}, 12)| =$$

Even terms

$1, 2, 4, 8, 16, 32, 64, 128$
Permutations Invariant Under $D$ and $D'$

Result

$$|B_{2k}^W(\bar{12}, 12)| = 2^k.$$  

Even terms

1, 2, 4, 8, 16, 32, 64, 128
Permutations Invariant Under $D$ and $\overline{D'}$

Result
\[ |B_{2^k}^W (\overline{12}, 12)| = 2^k. \]

Even terms
1, 2, 4, 8, 16, 32, 64, 128

Result
\[ |B_{2^k}^W (\overline{12}, \overline{12}, 12)| = \]

Even terms
1, 1, 2, 3, 6, 10, 20, 35
Permutations Invariant Under $D$ and $\overline{D'}$

**Result**

$|B_{2k}^W(\overline{12}, 12)| = 2^k$.

**Even terms**

1, 2, 4, 8, 16, 32, 64, 128

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**Result**

$|B_{2k}^W(\overline{12}, \overline{12}, 12)| = \binom{k}{\lfloor k/2 \rfloor}$.

**Even terms**

1, 1, 2, 3, 6, 10, 20, 35
Permutations Invariant Under $\overline{D}$

Result

$|B_{2k}^D(\overline{21}, \overline{21}, 21)| = C_k$

Even terms

1, 1, 2, 5, 14, 42, 132, 429
Result

$$|B_{2k}^{D}(\vec{1}, \vec{21}, 21)| = C_k.$$
Open Questions

- Combinatorial proofs:
  - \( |B_{2k}^{1\bar{0}}(\bar{21}, 21, 21)| = \binom{2k+1}{k} \)
  - \( |B_{2k}^{\bar{W}}(\bar{12}, 12, 12)| = \binom{k}{\lfloor k/2 \rfloor} \)

- \( r \)-colored permutations for \( r > 2 \).

- Avoidances of length 2 and length 3
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