The Combinatorics of Symmetric Functions: (3 + 1)-free Posets and
the Poset Chain Conjecture

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Abstract

There are a multitude of ways to generate symmetric functions, many of which have been described previously [4]. In 1995 Richard Stanley described a method for generating a symmetric function from the set of proper colorings of a simple graph [7]. This is known as the chromatic symmetric function and has raised a number of interesting questions, particularly regarding its expansion in terms of known bases of the vector space of symmetric functions. For instance, characterization of the e-coefficients (coefficients of the elementary symmetric function expansion) is an open problem in the field. Stanley published a conjecture (termed the Poset Chain Conjecture) which states that the e-coefficients of the chromatic symmetric function generated by the incomparability graph of a (3+1)-free poset are nonnegative [7]. This conjecture is still unproven. Here, we present results from two papers by Vesselin Gasharov and Timothy Chow which provide supporting evidence for the Poset Chain Conjecture, as well as some combinatorial insights that may aid in the search for a proof [5, 2].

1 Background

We have seen several bases for the vector space of symmetric functions, among them, the elementary symmetric functions and the Schur functions. Both types of functions can be described by fillings of Ferrers diagrams.

If $T$ is a filling of a Ferrers diagram with $wt(T) = (w_1, w_2, \ldots)$ then we write $x^T = \prod_i x_i^{w_i}$.

Definition 1.1. For any partition $\lambda$ we define the elementary symmetric function $e_\lambda = \sum_T x^T$ where the sum is over row strictly increasing fillings of the Ferrers diagram of $\lambda$ [4].

\[
\begin{array}{ccc}
3 & 4 \\
2 & 3 & 4 \\
4 & 5 & 6 & 8
\end{array}
\]

Figure 1: A row strictly increasing filling of the Ferrers diagram of 4,3,2. This contributes a term of $x_2x_3x_4x_5x_6x_8$ to $e_{4,3,2}$. 
Definition 1.2. For any partition $\lambda$ we define the Schur function $s_{\lambda} = \sum T x^T$ where the sum is over column strictly increasing, row nondecreasing fillings of the Ferrers diagram of $\lambda$. Such fillings are called semistandard tableaux or SST [4].

\[
\begin{array}{ccc}
3 & 4 \\
2 & 3 & 4 \\
1 & 1 & 6 \\
\end{array}
\]

Figure 2: A semistandard tableau of shape 3,3,2. This contributes a term of $x_1^3 x_2^4 x_3^2 x_4^2 x_6^1$ to $s_{3,3,2}$.

In this section, we will look at another type of symmetric function, the chromatic symmetric function. The chromatic symmetric function is a symmetric function which is created from a graph, so we begin with some definitions concerning graphs.

Definition 1.3. Given a graph $G$ with vertex set $V = V(G) = \{v_1, v_2, ..., v_d\}$, a coloring of $G$ is a function $\kappa: V \to P$, where $P = \{1, 2, ..., d\}$ is the set of colors.

Definition 1.4. A proper coloring is a coloring $\kappa$ such that if $u$ and $v$ are vertices connected by an edge, then $\kappa(u) \neq \kappa(v)$.

Example 1.5. Here are two colorings of the same graph. The first is an improper coloring because the middle two vertices are connected by an edge but have the same color. The second is a proper coloring because no two adjacent vertices share the same color:

Before we define the chromatic symmetric function, we define a related invariant of a graph: the chromatic polynomial.

Definition 1.6. The chromatic polynomial $X_G(n)$ of a graph $G$ is a polynomial such that $X_G(n)$ is the number of proper colorings of $G$ with $n$ colors [7].

Example 1.7. Consider the graph $P_3$ (the path of length 3).

\[
\begin{array}{c|ccc}
\hline
n & 0 & 1 & 2 & 3 \\
\hline
X_G(n) & 0 & 0 & 2 & 12 \\
\hline
\end{array}
\]

Below are the 2 proper colorings of a path of length 3 with 2 colors:

We can now use the chromatic polynomial to help us define the chromatic symmetric function.
Definition 1.8. The chromatic symmetric function for a graph $G$ is

$$X_G = X_G(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)}x_{\kappa(v_2)}\ldots x_{\kappa(v_d)} \quad (1)$$

where the sum is over all proper colorings of the graph [7].

All that this means is that for each proper coloring of a graph, we take the color of each vertex, put that color into the subscript of an $x$, and multiply all of those $x$'s together. We then add together the terms generated by all proper colorings.

The chromatic symmetric function is symmetric because we sum over all proper colorings. This sum accounts for all possible permutations, and thus our function is invariant under any permutation and is therefore symmetric.

Though it may not be immediately clear, $X_G$ does not uniquely determine $G$ as it does not distinguish between all graphs that are nonisomorphic. Stanley showed that $X_G$ can distinguish all nonisomorphic graphs with up to 4 vertices, but after that point there are pairs of nonisomorphic graphs that have the same chromatic symmetric function [7]. Figure 3 shows a pair of nonisomorphic graphs that have the same chromatic symmetric function.

![Figure 3: The chromatic symmetric function of each of these graphs is $X_G = 2m_{2,2,1} + 4m_{2,1,1,1} + m_{1,1,1,1,1}$ even though they are nonisomorphic. Note that $m$ represents the monomial symmetric function, another standard basis for the vector space of symmetric functions.](image)

Proposition 1.9. If $G$ and $H$ are disjoint graphs, and $G + H$ is the disjoint union of the two, then $X_{G+H} = X_G X_H$ [7].

This follows immediately from our definition of the chromatic symmetric function. If we have the union of 2 disjoint graphs, then the way we color the vertices of one of the disjoint pieces is completely independent of the way we color the vertices of the other. Thus the chromatic symmetric function of the whole graph can be generated by multiplying the chromatic symmetric functions of each disjoint part.

We have now defined the chromatic symmetric function in addition to the chromatic polynomial, so it is natural to want to know the relationship between the two.


This should be somewhat intuitive because of the fact that the chromatic polynomial is the number of proper colorings, and the chromatic symmetric function sums over all proper colorings. Thus by plugging in 1 for all the variables in the chromatic symmetric function, we get 1 per term, which is 1 per proper coloring, and thus the sum counts the number of proper colorings.

Now that we have a definition of the chromatic symmetric function, it is interesting to look at how $X_G$ can be written in terms of some of the common bases of the symmetric functions. This is precisely what Richard Stanley began to do, and he came up with the following (still unproven) conjecture.

Conjecture 1.11. The Poset Chain Conjecture: If $P$ is a $(3 + 1)$-free partially ordered set, and $inc(P)$ is the incomparability graph for $P$, then $X_{inc(P)}$ is e-positive [7].

This conjecture has yet to be proven, but progress has been made since it was first published in 1995. Most of the rest of this paper will be dedicated to outlining some of that progress, but first it is important that we understand exactly what the conjecture is saying as many of the terms used are probably unfamiliar.
Recall that a partially ordered set (or poset) is a set together with a binary relation that indicates that, for certain pairs of elements in the set, one of the elements precedes the other. It is important to note that not all elements need be comparable. For example, we can define a partial ordering \(|\) on the integers such that for all \(a, b \in \mathbb{Z}, a|b\) if \(a\) is a factor of \(b\). So \(4|20\) and \(5|20\) but \(4 \not| 5\).

**Definition 1.12.** Given a poset \(P\), the *incomparability graph* of \(P\), written \(inc(P)\), is a graph with every element of the poset as a vertex, and edges between vertices if the elements are incomparable under the partial ordering. A *comparability graph* is defined similarly, except edges exist in the graph if the elements of the poset are comparable.

It should be clear from this definition that a poset’s comparability graph and incomparability graph are complements of one another.

**Definition 1.13.** We say that a poset \(P\) is \((3 + 1)\)-free if \(P\) contains no induced subposet that is the union of disjoint 3-element and 1-element chains.

This means that no matter what 4 elements we choose from our partial ordering, we will never get 3 that are in a chain, along with 1 that is not comparable to any of the other 3. If we want to visualize this, it means no induced subgraph of the comparability graph looks like:

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  |
  |
  |
  |
  |
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The final key definition is of \(e\)-positivity.

**Definition 1.14.** A symmetric function is said to be \(e\)-positive whenever it can be written as a sum of the elementary symmetric functions such that all of the coefficients are non-negative.

Note that it is always possible to write a symmetric function as a unique combination of elementary symmetric functions because they form a basis for the vector space of symmetric functions.

We should now have a good understanding of the Poset Chain Conjecture. In essence, the conjecture says that, given a \((3 + 1)\)-free poset, the chromatic symmetric function of the incomparability graph of the poset can be written as a linear combination of elementary symmetric functions with non-negative coefficients. For the sake of brevity, from this point on when we say “the graph is \(e\)-positive” or something similar, we mean “the chromatic symmetric function of the graph is \(e\)-positive”. With this understanding in mind we can move on to look at some of the work that has been done in trying to prove Stanley’s conjecture.

## 2 Gasharov’s Paper on \(s\)-positivity

We have now introduced both Ferrers diagrams and graph colorings as means to generate symmetric functions. It’s natural to ask how these different combinatorial objects relate to one another. Vesselin Gasharov developed a correspondence between colorings of graphs and fillings of Ferrers diagrams in order to prove that incomparability graphs of \((3+1)\)-free posets are \(s\)-positive (that is, the chromatic symmetric function of the incomparability graph of a \((3+1)\)-free poset can be written as the sum of Schur functions with non-negative coefficients) \([5]\). In this section we describe Gasharov’s work.

### 2.1 Setup and Definitions

The following theorem is the main result of Gasharov’s work. The rest of this section will be spent outlining Gasharov’s proof.

**Theorem 2.1.** If \(G\) is the incomparability graph of a \((3+1)\)-free poset, then \(G\) is \(s\)-positive.
To prove this theorem, Gasharov first finds a way to relate colorings of incomparability graphs to fillings of Ferrers diagram-like arrays. These fillings are then be restricted to objects reminiscent of semistandard tableaux.

To begin, we extend the idea of colorings in order to work in greater generality.

**Definition 2.2.** Let $G$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_d\}$. A **multicoloring** of $G$ is a function $\kappa : V \to 2^P$, where $2^P$ is the power set of $P = \{1, 2, \ldots\}$.

We can think of a multicoloring as a coloring where each vertex is given a (finite) flag of colors, rather than just a single color. A multicoloring is **proper** when no two adjacent vertices have any colors in common on their flags. More formally, $\kappa$ is proper if $\kappa(u) \cap \kappa(v) = \emptyset$ whenever $u$ and $v$ share an edge. This is obviously a generalization of the standard notion of a coloring; every coloring is a multicoloring in which each vertex has only one color in its flag. Given a sequence of nonnegative integers $m = (m_1, m_2, \ldots, m_d)$, we can define an $m$-**multicoloring** $\kappa$ such that $|\kappa(v_i)| = m_i$. So a standard coloring is a $(1,1,\ldots,1)$-multicoloring. Just as we defined $x^T$ as the product over the elements of a filling of a tableau, for a finite subset $S = \{s_1, s_2, \ldots\}$ we define $x_S = x_{s_1}x_{s_2}\ldots$, the product over elements of the set. This allows us to create the following natural generalization of the chromatic symmetric function.

**Definition 2.3.** Let $m = (m_1, m_2, \ldots, m_d)$.

$$X_G^m = X_G^m(x_1, x_2\ldots) = \sum_{\kappa} x_{\kappa(v_1)}x_{\kappa(v_2)}\cdots x_{\kappa(v_d)}$$

(2)

where the sum on the right is over all proper $m$-multicolorings of $G$.

**Example 2.4.** Let $m = (2, 2, 2, 2)$ and $G = K_4$. Each term of $X_G^m$ will have degree 8, because there are 8 total colors in the graph. Moreover, no variable will have degree greater than 1 because no two vertices can share a color. So $X_G^m = p \cdot e_8$ where $p$ is a constant. We can assign 8 colors by choosing 2, for the first vertex, then 2 for each subsequent vertex. So we get: $p = \binom{8}{2}\binom{6}{2}\binom{4}{2} = \binom{8}{2,2,2,2}$. This gives us $X_G^m = \binom{8}{2,2,2,2}e_8$. In general, if $G = K_4$ and the elements of $m$ sum to $n$, then $X_G^m = \binom{m}{m_1,m_2,\ldots,m_d}e_n$.

So to prove that $X_G$ is $s$-positive, it is sufficient to show that $X_G^m$ is $s$-positive.

**Theorem 2.5.** If $G$ is the incomparability graph of a $(3+1)$-free poset with $d$ elements, and $m = (m_1, m_2, \ldots, m_d)$ is any sequence of nonnegative integers, then $X_G^m$ is $s$-positive.

We now provide a sketch of the proof. First, define a set of Ferrers diagram-like objects that can be put into bijection with multicolorings. Next, assign each of these objects a sign (either 1 or -1). Finally, construct an involution that cancels certain elements, such that the elements we are left with have only positive signs.

**Definition 2.6.** Let $P = (P, \leq)$ be a partially ordered set. A $P$-**array** is an array of elements of $P$, arranged in bottom-justified rows, such that columns are strictly increasing.

![Figure 4](image-url) A P-array of shape $(5, 3, 4, 0, 2)$ for the poset $(\mathbb{Z}, \mid)$. Note that gaps between columns are allowed and no restrictions are placed on the elements in any row.
In this definition, elements can appear in an arbitrary number of columns, but they can appear at most once per column. We are also using an arbitrary poset, so we are generalizing an idea that we have seen before: every column strict filling of a Ferrers diagram with positive integers is a P-array with \( P = (\mathbb{Z}, \leq) \). Finally, every element in a given column is comparable to every other elements in that column (by transitivity). We say that a P-array has shape \( \lambda \) when column \( i \) has \( \lambda_i \) elements. This is the transpose of our normal sense of shape (where we list the lengths or rows). We need to use this definition because we can have rows with gaps, but not columns with gaps in a P-array.

We now refine the idea of a P-array with the definition of a P-tableau.

**Definition 2.7.** A P-tableau is a P-array with two additional conditions: adjacent elements in the same row are nondecreasing from left to right (including not comparable) and the height of columns are weakly decreasing.

\[
\begin{array}{cccc}
32 \\
16 & 40 \\
8 & 20 & 80 \\
4 & 10 & 20 & 24 & 24 \\
2 & 5 & 5 & 8 & 8 & 11
\end{array}
\]

Figure 5: A P-tableau for the poset \( (\mathbb{Z}, |) \). Note that the shape is that of a partition and each row is nondecreasing from left to right.

The weight of a P-array (or P-tableau) is \( (n_1, n_2, \ldots) \), where \( n_i \) is the number of occurrences of \( v_i \) in \( T \). With P-tableaux, we have objects that look exactly like Ferrers diagrams, but whose elements are members of a poset, rather than the integers. Moreover, when \( P = (\mathbb{Z}, \leq) \), P-tableaux are exactly semistandard tableaux. Note that a P-tableau of shape \( \lambda' \) looks like a Ferrers diagram of shape \( \lambda \).

Now that we have defined some useful objects, let’s look at their relationship to graph colorings. Let \( G \) be an incomparability graph of \( (P, \leq) \). Suppose we have a proper multicoloring \( \kappa \); then we can associate \( \kappa \) with a P-array \( T_\kappa \). For all colors \( i \), let \( \kappa^{-1}(i) = \{ v_1^{(i)}, v_2^{(i)}, \ldots \} \). More informally \( \kappa^{-1}(i) \) is the set of vertices (elements of \( P \)) that have color \( i \) in \( \kappa \). Since \( \kappa \) is proper and \( G \) is an incomparability graph, if two vertices have the same color then they must be comparable. If they were not comparable, an edge would exist in \( G \), and therefore \( \kappa \) would not be proper. So we can order the elements of \( \kappa^{-1}(i) \) from smallest to largest to generate our P-array.

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
v_2^{(1)} & v_2^{(2)} & \ldots \\
v_1^{(1)} & v_1^{(2)} & \ldots \\
\end{array}
\]

Figure 6: The P-array associated with a multicoloring \( \kappa \).

Given a P-array \( T \), we can define a unique multicoloring \( \kappa \) such that \( T = T_\kappa \). To do this we look at every column in which element \( j \) appears and give vertex \( v_j \) those colors. Thus \( \kappa(v_j) = \{ i | j \text{ appears in column } i \} \). This will generate a proper multicoloring for reasons similar to those just described.

**Example 2.8.** Define \( | \) as the poset generated by divisibility such that \( i | j \) if \( i \) is a factor of \( j \). Let \( P = \{ (2, 3, 4, 5, 8, 9, 10), | \} \). Let \( \kappa = \{ (2, \{ 1 \}), (3, \{ 6 \}), (4, \{ 1, 3 \}), (5, \{ 2 \}), (8, \{ 1, 4 \}), (9, \{ 6 \}), (10, \{ 2 \}) \} \). Then \( \kappa \) is a proper coloring of \( \text{inc}(P) \), and \( T_\kappa \) is its corresponding P-array.
2.2 Gasharov’s Proof

To prove $s$-positivity of $X^m_{inc(P)}$, Gasharov shows that for any partition $\lambda$, the coefficient of $s_\lambda$ counts $P$-tableaux of shape $\lambda$ and weight $m$. Because the coefficient of $s_\lambda$ counts some set of objects, it must be nonnegative.

The Schur functions are a basis for the vector space $\Lambda$ of symmetric functions. The inner product

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$$

has the property $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$ for all partitions $\lambda$ and $\mu$.

Because the Schur functions are a basis for the vector space of symmetric functions, we can uniquely write $X^m_G$ as a linear combination of Schur functions, so $X^m_G = \sum c_\mu s_\mu$. Applying the inner product $\langle -, - \rangle$,

$$\langle X^m_G, s_\lambda \rangle = \langle \sum c_\mu s_\mu, s_\lambda \rangle = \sum c_\mu \langle s_\mu, s_\lambda \rangle = c_\lambda \quad (3)$$

The first Jacobi-Trudi identity allows us to change bases between the Schur functions and the complete homogeneous symmetric functions (see appendix B). The first Jacobi-Trudi identity says

$$s_\lambda = \det_{i,j} H^j_{\lambda_i}$$

where $H_{i,j} = h_{\lambda_i - i + j}$. It is a fact that for any matrix $A$, $\det A = \sum_{\pi \in S_{\ell(\lambda)}} sgn(\pi) \prod_{j=1}^{\ell(\lambda)} A_{\pi(j),j}$. For $\pi \in S_{\ell(\lambda)}$ we define

$$\pi(\lambda) = \{\lambda_{\pi(j)} - \pi(j) + j\}_{j=1}^{\ell(\lambda)}$$

which allows us to write

$$s_\lambda = \sum_{\pi \in S_{\ell(\lambda)}} sgn(\pi) \prod_{j=1}^{\ell(\lambda)} H_{\pi(j),j} = \sum_{\pi \in S_{\ell(\lambda)}} sgn(\pi) h_{\pi(\lambda)}. \quad (4)$$

Putting (3) and (4) together, we get

$$c_\lambda = \langle X^m_G, s_\lambda \rangle = \langle \sum_{\pi \in S_{\ell(\lambda)}} sgn(\pi) h_{\pi(\lambda)}, h_{\pi(\lambda)} \rangle = \sum_{\pi \in S_{\ell(\lambda)}} sgn(\pi) \langle X^m_G, h_{\pi(\lambda)} \rangle. \quad (5)$$

Then, we write $X^m_G = \sum_{\lambda} a_\lambda m_\lambda$ and take the inner product

$$\langle X^m_G, h_\lambda \rangle = \langle \sum_{\mu} a_\mu m_\mu, h_\lambda \rangle \sum_{\mu} \langle m_\mu, h_\lambda \rangle = a_\lambda. \quad (6)$$
Combining (5) and (6) gives us \( \langle X_G^m, h_{\pi(\lambda)} \rangle = a_{\pi(\lambda)} \). By the definition of the monomial symmetric functions, \( a_{\pi(\lambda)} \) is the coefficient of of the term \( x_1^{\pi(\lambda)_1} x_2^{\pi(\lambda)_2} \cdots x_{l(\lambda)}^{\pi(\lambda)_{l(\lambda)}} \), which is the number of \( m \)-multicolorings of \( G \) using color \( i \) exactly \( \pi(\lambda)_i \) times. Each such multicoloring corresponds to a \( P \)-array with shape \( \pi(\lambda) \) because the \( i \)th column has \( \pi(\lambda)_i \) entries. Therefore, \( \langle X_G^m, h_{\pi(\lambda)} \rangle \) is the number of \( P \)-arrays of shape \( \pi(\lambda) \) and weight \( m \). From (5) we have \( \lambda \in S_{l(\lambda)} \)
\[
c_\lambda = \sum_{\pi \in S_{l(\lambda)}} sgn(\pi)\langle X_G^m, h_{\pi(\lambda)} \rangle.
\]
Let \( A = \{ (\pi, T) | \pi \in S_{l(\lambda)} \text{ and } T \text{ is a } P \text{-array of shape } \pi(\lambda) \text{ and weight } m \} \). We can rewrite (7) as
\[
c_\lambda = \sum_{(\pi, T) \in A} sgn(\pi).
\]

**Lemma 2.9.** If \( T \) is a \( P \)-tableau of shape \( \pi(\lambda) \) then \( \pi \) is the identity permutation \( \text{id} = 12\ldots l(\lambda) \).

**Proof.** Suppose \( T \) is a \( P \)-tableau of shape \( \pi(\lambda) \); it must be the case that \( \pi(\lambda)_1 \geq \pi(\lambda)_2 \geq \ldots \geq \pi(\lambda)_{l(\lambda)} \).
Suppose \( \pi \neq \text{id} \). Then \( \pi \) must have at least one inversion such that \( i < j \) and \( \pi(i) > \pi(j) \). Since \( \lambda_a \geq \lambda_b \) whenever \( a < b \), it must be the case that \( \lambda_{\pi(i)} \leq \lambda_{\pi(j)} \Rightarrow \lambda_{\pi(i)} - \pi(i) < \lambda_{\pi(j)} - \pi(j) \Rightarrow \lambda_{\pi(i)} - \pi(i) + i < \lambda_{\pi(j)} - \pi(j) + j \Rightarrow \pi(\lambda)_i < \pi(\lambda)_j \), which violates our assumption. Therefore \( T \) cannot be a \( P \)-tableau. \( \square \)

Notice that the identity permutation has no inversions, therefore \( sgn(\text{id}) = 1 \). Given (5), if the sum
\[
\sum_{\pi \in S_{l(\lambda)}} \sum_{T \in B} sgn(\pi)
\]
could be restricted to \( P \)-tableau, then \( c_\lambda \) would be nonnegative for all partitions \( \lambda \), proving that \( X_G^m \) is \( s \)-positive.

Let \( B = \{ (\pi, T) | \pi \in S_{l(\lambda)} \text{ and } T \text{ is a } P \text{-array of shape } \pi(\lambda) \text{ and weight } m \text{ and } T \text{ is not a } P \text{-tableau} \} \). If we can find an involution \( \phi : B \rightarrow B \) such that \( \phi(\pi, T) = (\sigma, T') \Rightarrow sgn(\sigma) = -sgn(\pi) \), then
\[
\sum_{(\pi, T) \in B} sgn(\pi) = 0
\]
so that
\[
c_\lambda = \sum_{(\pi, T) \in A \setminus B} sgn(\pi).
\]

We describe one such involution \( \phi \) below.

Let \( T \) be a \( P \)-array of shape \( \pi(\lambda) \) (for some \( \pi \in S_{l(\lambda)} \)) and weight \( m \). If \( T \) is not a \( P \)-tableau, then there exists at least one entry \( a_{ij} \in T \) such that the condition for a \( P \)-tableau fails at \( a_{ij} \) (that is, \( a_{i,j+1} \) is defined and \( a_{i,j} \) is undefined or \( a_{i,j} < a_{i,j+1} \)). Let \( r = r(T) \) be the smallest positive integer such that the condition for a \( P \)-tableau fails at \( a_{r_c} \) for some \( i \). Let \( c = c(T) \) be the largest positive integer such that the condition for a \( P \)-tableau fails at \( a_{r_c} \).

Define \( \phi : B \rightarrow B \) such that \( \phi(\pi, T) = (\sigma, T') \), where \( \sigma = \pi \circ (r, r+1) \), and \( T' \) is the array obtained from \( T \) by switching column \( c \) from row \( r \) on with column \( c+1 \) from row \( r+1 \) on.

For example, if the \( 5 \times 4 \) section around \( a_{r,c} \) in \( T \) looks like

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
 a_{r+2,c-1} & a_{r+2,c} & a_{r+2,c+1} & a_{r+2,c+2} \\
 a_{r+1,c-1} & a_{r+1,c} & a_{r+1,c+1} & a_{r+1,c+2} \\
 a_{r,c-1} & a_{r,c} & a_{r,c+1} & a_{r,c+2} \\
 a_{r-1,c-1} & a_{r-1,c} & a_{r-1,c+1} & a_{r-1,c+2} \\
 a_{r+1,c-1} & a_{r+1,c} & a_{r+1,c+1} & a_{r+1,c+2} \\
 a_{r+2,c-1} & a_{r+2,c} & a_{r+2,c+1} & a_{r+2,c+2} \\
 a_{r,c-1} & a_{r,c} & a_{r,c+1} & a_{r,c+2} \\
 a_{r-1,c-1} & a_{r-1,c} & a_{r-1,c+1} & a_{r-1,c+2} \\
 a_{r+1,c-1} & a_{r+1,c} & a_{r+1,c+1} & a_{r+1,c+2} \\
 \vdots & \vdots & \vdots & \vdots \\
 \end{array}
\]
Recall that if $\phi(\pi, T) = (\sigma, T')$ then $\sigma = \pi \circ (r, r+1)$. The height of column $c+1$ in $T$ is $\pi(\lambda)c+1$, therefore the height of column $c$ in $T'$ is $\pi(\lambda)c+1 - 1 = \lambda_{\pi(c+1)} - \pi(c+1) + (c+1) - 1 = \lambda_{\pi(c)} - \sigma(c) + c = \sigma(\lambda)c$. Similarly, the height of column $c$ in $T$ is $\pi(\lambda)c$, so the height of column $c+1$ in $T'$ is $\pi(\lambda)c + 1 = \lambda'_{\pi(c)} - \pi(c) + c + 1 = \lambda_{\sigma(c+1)} - \sigma(c) + c + 1 = \sigma(\lambda)c+1$. For $i \neq c, c+1$ column $i$ has height $\pi(\lambda)i = \sigma(\lambda)i$, so $T'$ has shape $\sigma(\lambda)$.

It is next necessary to check that $T'$ is actually a $P$-array. In order to be a $P$-array, it must be the case that $a_{ij} \prec a_{i+1,j}$ whenever $a_{ij}$ and $a_{i+1,j}$ are both defined. The only pairs of elements we have to check, however, are those that were not already adjacent in $T$. One such pair is found in column $c+1$ of $T'$, with $a_{r,c+1}$ directly below $a_{r,c}$. Notice that in $T$, $a_{r,c}$ was to the left of $a_{r,c+1}$ and recall that the condition for a $P$-tableau failed in $a_{r,c}$. Therefore $a_{r,c+1}$ is defined and either $a_{r,c}$ is not defined or $a_{r,c} \succ a_{r,c+1}$, either case satisfies the condition for a $P$-array.

The other pair of newly adjacent elements in $T'$ is in column $c$ with $a_{r-1,c}$ directly below $a_{r-1,c+1}$. Since $T$ is a $P$-array, it must be the case that $a_{r-1,c} < a_{r-1,c+1}$ as long as all three are defined (note that $a_{r,c+1}$ is defined by assumption and $a_{r-1,c+1}$ must be defined because we don’t have any spaces within columns of a $P$-array, and if $a_{r-1,c+1}$ is not defined we do not violate the condition for a $P$-array in $T'$). Assuming all three are defined, they form a chain in $P$. Since $P$ is $(3+1)$-free by assumption, $a_{r-1,c-1}$ must be comparable to one of the three elements of the chain. Because row $r$ is the lowest row where the condition of a $P$-tableau fails, $a_{r-1,c} \not< a_{r-1,c+1} \Rightarrow a_{r-1,c} \not< a_{r-1,c+1} \Rightarrow a_{r-1,c} \not< a_{r-1,c+1}$. Therefore, $a_{r-1,c} < a_{r-1,c+1}$ or $a_{r-1,c} < a_{r,c+1}$ or $a_{r-1,c} < a_{r+1,c-1}$: in any case, transitivity gives us $a_{r-1,c} < a_{r+1,c+1}$. $T'$ is therefore a $P$-array.

$T'$ is not a $P$-tableau; in fact, the condition for a $P$-tableau fails in the same place. The element in row $r$ and column $c$ of $T'$ is $a_{r+1,c+1}$ and the element immediately to the right is $a_{r,c+1}$. $a_{r,c+1}$ exists by our initial assumptions, and since $T$ is a $P$-array $a_{r,c} < a_{r,c+1}$ or else $a_{r+1,c+1}$ does not exist. In either case, the condition for a $P$-tableau fails at row $r$ and column $c$. Notice that $\phi$ does not change any rows below row $r$, and row $r$ is not changed to the right of column $c$, so row $r$, column $c$ is still the lowermost and rightmost place where the $P$-tableau condition fails. This means that $\phi$ is an involution, since all elements will be moved back to their original locations when $\phi$ is applied to $T'$.

Summarizing the above, $\phi$ maps a non-tableau $P$-array of shape $\pi(\lambda)$ to a non-tableau $P$-array of shape $\sigma(\lambda)$ where $\sigma = \pi \circ (r, r+1)$. Note that since $\sigma$ has one more or one less inversion than $\pi$, $sgn(\sigma) = -sgn(\pi)$. Thus, $\phi$ gives a one-to-one correspondence between elements of $B$ with positive and negative sign. Therefore

$$
\sum_{(\pi,T) \in B} sgn(\pi) = 0
$$

which gives us

$$
c_\lambda = \sum_{(\pi,T) \in A\setminus B} sgn(\pi)
$$

as we hoped. Notice that

$$
A\setminus B = \{(\pi,T) | \pi \in S_{\ell(\lambda)} \text{ and } T \text{ is a } P\text{-tableau of shape } \pi(\lambda) \text{ and weight } m\}.
$$

By Lemma 2.9, if $T$ is a $P$-tableau then $\pi(i) = i$ for $i = 1, ..., \ell(\lambda)$, so

$$
\pi(\lambda) = \{\lambda_{\pi(j)} - \pi(j) + j\}_{j=1}^{\ell(\lambda)} = \{\lambda_{\pi(j)}\}_{j=1}^{\ell(\lambda)} = \lambda.
$$
As mentioned previously, $\text{sgn}(\text{id}) = 1$, so

$$c_\lambda = \sum_{(\pi, T) \in A \setminus B} \text{sgn}(\pi) = \sum_{T \in C} 1$$

(8)

where $C = \{T | T \text{ is a } P\text{-tableau of shape } \lambda \text{ and weight } m\}$.

In other words, (8) shows that $c_\lambda$ is the number of $P$-tableaux of shape $\lambda$ and weight $m$. Therefore, if $G$ is the incomparability graph of a $(3 + 1)$-free poset, then $X^m_G = \sum_{\mu} c_\mu s_\mu$ where $c_\lambda \geq 0$ for all $\lambda$.

**Example 2.10.** Define $\sim_{\pi}$ as the poset generated by a permutation $\pi$ such that $i \sim_{\pi} j$ if $i$ and $j$ occur in order in $\pi$. This means that $i \sim_{\pi} j$ whenever $i \leq j$ and $\pi(i) \leq \pi(j)$. Let $P = (\{1, 2, 3, 4, 5, 6, 7, 8\}, \sim_{\pi})$ and let $\pi = 12365487$. $P$ is $(3 + 1)$-free (see appendix A). Figure 8 shows two legal $P$-arrays. Note that $5 \not\sim_{\pi} 3$

$$\begin{array}{cccc}
8 & 3 & 7 & 4 \\
2 & 5 & 3 & 8 \\
1 & 2 & 2 & 6 \\
\end{array}$$

$$\begin{array}{cccc}
8 & 3 & 7 & 5 \\
2 & 4 & 3 & 8 \\
1 & 2 & 2 & 6 \\
\end{array}$$

Figure 8: Two $P$-arrays.

since 3 occurs before 5 in the permutation. Additionally, this is the farthest right column in the first row in which the $P$-array fails to obey the $P$-tableau criterion. Thus, these two $P$-arrays are mapped to each other by Gasharov’s involution. Moreover, they have opposite signs. The left has $\pi = 1234$ with $\text{sgn}(\pi) = 1$ and the right has $\sigma = 1324$ with $\text{sgn}(\sigma) = -1$.

See [1] for a more in-depth treatment of similar posets.

3 Chow’s Combinatorial Interpretation of the e-Coefficients of the Chromatic Symmetric Function

Gasharov’s result, apart from giving “supporting evidence” for the poset chain conjecture, is useful because it essentially breaks the problem of proving the conjecture into smaller parts. There is already a significant body of research on methods for changing basis in the ring of symmetric functions, so by getting the chromatic symmetric function in terms of one well-known basis, it is much simpler to write the function in terms of another specific basis, in this case the elementary symmetric functions.

Chow [2] uses a previous result of Egecioglu and Remmel [3] to get an explicit expression for the $e$-coefficients, as well as a corresponding combinatorial interpretation, given Gasharov’s result. Unfortunately, Chow’s result includes a signed sum, meaning that it is not by itself sufficient to prove the poset chain conjecture. Chow expresses hope, however, that a sign-reversing involution on this sum will give a combinatorial proof of the poset chain conjecture, and in support of this hope, gives a sign-reversing involution for certain sums of the $e$-coefficients that relate to acyclic orientations of graphs.

3.1 Combinatorics of the Inverse Kostka Matrix

There will be quite a few definitions and previous results we must present before reaching Chow’s main result, as his main result follows from a fairly straightforward application of the work of Gasharov and others. For the purposes of this section, we will assume that all graphs $G$ to which we refer are the incomparability graphs for $(3 + 1)$-free posets.

The first of the results we investigate is that of Egecioglu and Remmel on the inverse Kostka matrix. The Kostka matrix, denoted $K$, has the property that its entries, indexed by partitions $\lambda$ and $\mu$, are given
by \( K_{\lambda,\mu} = |\text{SST}_\mu(\lambda)| \) where \( \text{SST}_\mu(\lambda) \) denotes the set of semistandard Young tableau of shape \( \lambda \) and content \( \mu \). Additionally, the Kostka matrix has the property that

\[
s_\lambda = \sum_{\mu \vdash |\lambda|} K_{\lambda,\mu} m_\mu
\]

where \( s_\lambda \) denotes the Schur function given by the partition \( \lambda \) and \( m_\mu \) is the monomial symmetric function corresponding to the partition \( \mu \). In essence, this means that the Kostka matrix acts as the change of basis matrix between \( s \) and \( m \).

Similarly, the inverse Kostka matrix \( K^{-1} \) acts as a change of basis matrix between Schur functions and the set of \( h \), the complete homogeneous symmetric functions. Namely,

\[
s_\mu = \sum_{\lambda \vdash |\mu|} K^{-1}_{\lambda,\mu} h_\lambda
\]

Simply by using the above facts and algebra, we can get the Kostka and inverse Kostka matrices.

Example 3.1. The matrices \( K \) and \( K^{-1} \) for partitions of size 4 are given above.

However, the algebraic definition of \( K^{-1} \) is both unwieldy and not particularly useful for our purposes, as its use requires calculation of the entire matrix and it gives us no combinatorial insights. Egecioglu and Remmel [3] introduce a type of signed combinatorial object and show that the entries of \( K^{-1} \) are sums over the signs of those objects. The full proof of this fact is beyond the scope of our project, but we will introduce the combinatorial objects and give a brief overview of Egecioglu and Remmel’s proof. We begin with rim hooks.

Definition 3.2. A rim hook is a collection of cells in a Ferrers diagram such that:

1. The cells lie on a connected path through the Ferrers diagram consisting only of southern and eastern steps.
2. When the cells are removed, the remaining cells in the diagram still form a legal Ferrers diagram.

A rim hook is called special if it contains at least one cell in the leftmost column of the Ferrers diagram. The sign of a rim hook is given by \((-1)^h\) where \( h \), the height of the rim hook, is the number of rows in which the rim hook has at least one element.

We introduce a notion of decomposing a Ferrers diagram into special rim hooks in the following way.

Definition 3.3. A special rim hook tabloid of shape \( \mu \) and type \( \lambda \) is a collection of connected simple paths of cells in the Ferrers diagram of shape \( \mu \) such that:

1. Each path consists of only southern and eastern steps.
2. Each path contains at least one element in the leftmost column.
3. Each cell is in exactly one path.
4. If we list the number of cells in each path in nonincreasing order, we get the partition \( \lambda \).
Note that this definition is equivalent to starting with the shape \( \mu \), then recursively removing special rim hooks from the shape until there are no cells left in the Ferrers diagram. In particular, note that each path \( \rho \) given by the tabloid is itself a special rim hook of the Ferrers diagram resultant from removing all the special rim hooks to the right of \( \rho \).

This notion allows us to define the sign of a special rim hook tabloid to be the product of the signs of its constituent rim hooks.

Egecioglu and Remmel [3] proved that for any partitions, \( \lambda, \mu \), we have,

\[
K_{\lambda,\mu}^{-1} = \sum_T \text{sgn} T
\]

where the sum on the right is over all special rim hook tabloids of type \( \lambda \) and shape \( \mu \).

Their proof involves using the Jacobi-Trudi identities to get a statement of \( s \) in terms of the \( h \) functions, then noting a relationship between those values and the height of rim hooks beginning in the first column. See [3] for a more thorough treatment of this proof.

We note that this statement is supported by our example \( K^{-1} \) above. In particular, nothing in the lower triangle can be non-zero, since at least 1 rim hook in any decomposition of the given type will not include an element of the first row, and, as expected, there is exactly 1 filling of type \( \mu \) for every partition \( \mu \), given by special rim hooks directly across the rows.

As a final note, recall that the inverse Kostka matrix gives the Schur functions in terms of complete homogeneous functions, not elementary symmetric functions. To resolve this issue, we use the involution \( \omega \), which is defined as the unique linear transformation such that

1. \( \omega(e_\lambda) = h_\lambda \)
2. \( \omega(h_\lambda) = e_\lambda \)
3. \( \omega(s_\lambda) = s_{\lambda'} \)

where \( \lambda' \) denotes the conjugate of the permutation \( \lambda \).

Since \( \omega \) is by definition linear, we know that

\[
s_\mu = \sum_{\lambda' \vdash |\mu|} K_{\lambda,\mu}^{-1} h_\lambda \Rightarrow \omega(s_\mu) = \sum_{\lambda' \vdash |\mu|} K_{\lambda,\mu}^{-1} \omega(h_\lambda) = \sum_{\lambda' \vdash |\mu|} K_{\lambda,\mu}^{-1} e_\lambda
\]

### 3.2 e-Coefficients Using Rim Hooks

Because \( \omega \) maps \( s_\lambda \) to \( s_{\lambda'} \), and Gasharov gives us that \( X_G = \sum_{\lambda' \vdash d} c_\lambda s_\lambda \) where \( c_\lambda \) denotes the number of \( P \)-tableaux of shape \( \lambda \), we can see that

\[
\omega(X_G) = \sum_{\lambda' \vdash d} c_{\lambda'} s_{\lambda'}. \]

The coefficient of \( s_\lambda \) in the above sum is \( c_{\lambda'} \), the number of \( P \)-tableaux of shape \( \lambda' \). We will say \( c_{\lambda'} = f_{\lambda'}^G \), so \( f_{\lambda'} \) denotes the number of \( P \)-tableaux of shape \( \lambda' \) (which look like Ferrers diagrams for \( \lambda \)).

The chromatic symmetric function can be written in terms of elementary symmetric functions as \( X_G = \sum_\lambda a_\lambda^G e_\lambda \), so we get:
\[
\sum_{\lambda} a_{\lambda}^{G} h_{\lambda} = \omega \left( \sum_{\lambda} a_{\lambda}^{G} e_{\lambda} \right) \\
= \omega (X_{G}) \\
= \sum_{\mu} f_{\mu}^{G} s_{\mu} \\
= \sum_{\mu} f_{\mu}^{G} \sum_{\lambda} K_{\lambda, \mu}^{-1} h_{\lambda} \\
= \sum_{\lambda} \left( \sum_{\mu} K_{\lambda, \mu}^{-1} f_{\mu}^{G} \right) h_{\lambda}.
\]

(1)

Because the left side of (1) and the right side of (2) both have a summation over complete homogeneous symmetric functions, indexed by the same partitions \( \lambda \), we can match up terms, and we get:

\[
a_{\lambda}^{G} = \sum_{\mu} K_{\lambda, \mu}^{-1} f_{\mu}^{G}
\]

and so we have written the \( e \) coefficients in terms of elements of the Kostka matrix, as we desired. This gives us an algebraic representation of the \( e \)-coefficients. However it is also combinatorial because we have a combinatorial interpretation for \( K \). In particular,

\[
a_{\lambda}^{G} = \sum_{\mu} \sum_{T} (\text{sgn } T) f_{\mu}^{G}
\]

where the sum is over special rim hook tabloids of shape \( \mu \) and type \( \lambda \). \[2\]

We can further note that \( f_{\mu} \) counts objects corresponding to a Ferrers diagram of shape \( \mu \), and \( T \) is constructed using a Ferrers diagram of shape \( \mu \), so we can, in essence, combine them.

**Definition 3.4.** A special rim hook \( P \)-tableau of shape \( \mu \) and type \( \lambda \) is a \( P \)-tableau of shape \( \mu \) combined with a special rim hook tabloid of shape \( \mu \) and type \( \lambda \) such that the cells are filled with elements of a poset according to the rules of \( P \)-tableaux, and there is an associated decomposition of the Ferrers diagram into rim hooks of type \( \lambda \).

We say the sign of a special rim hook \( P \)-tableau is the sign of its special rim hook tabloid, so as a result, we have a completely combinatorial way of writing the coefficients of \( e \); namely:

\[
a_{\lambda}^{G} = \sum_{\mu} \sum_{T} \text{sgn } T f_{\mu}^{G}
\]

where \( T \) is a special rim hook \( P \)-tableau of type \( \lambda \). \[2\]

This is Chow’s result on \( a_{\lambda} \); however it does not prove the poset chain conjecture since the sum is signed. His hope is to produce a sign-reversing involution on this sum to get a proof of the poset chain conjecture. In support of this hope, he concludes his paper with his own sign-reversing involution on this sum. This involution does not prove the poset chain conjecture, because it counts sums of \( e_{\lambda} \) coefficients, thus proving the positivity of certain sums of coefficients, but not of the individual coefficients in that sum.

### 3.3 Chow’s Sign Reversing Involution

We begin with a definition.
**Definition 3.5.** [2] An acyclic orientation of a graph $G$ is an assignment of a direction to each edge of $G$ in such a way that no directed cycles are formed. A sink of such an orientation is a vertex which has no outgoing edges.

Chow shows that if $G$ is the incomparability graph of a $(3+1)$-free poset $P$, then for all $k$,

$$
\sum_{\lambda \vdash (l(\lambda) = k)} a_{\lambda}^G
$$

where $G$ is the number of acyclic orientations of $G$ with exactly $k$ sinks.

The proof consists of finding an expression for the number of $P$-tableaux which are hooks, and then using the result of the previous section to cancel all special rim hook $P$-tableaux which are not hooks. He then finds a bijection between acyclic orientations of graphs with $k$ sinks and the special rim hook $P$-tableaux of hooks with type $\lambda$ where $l(\lambda) = k$, completing his proof. See [2] for more details.

## 4 Conclusion

Much work is left to do in order to finish the proof of the poset chain conjecture, but there do exist some clear avenues of exploration. As mentioned earlier, Chow left us with a combinatorial interpretation of the $e$ coefficients, however he sums over signed objects. If we could create a sign-reversing involution similar to Gasharov’s to cancel out the negative objects, the conjecture would be proven. Though promising, this method has yet to yield a full proof. See Sagan and Lee [6] for further reading on the most recent progress.

One other avenue to explore is to use posets generated by permutations to give more supporting evidence. These do not represent all posets, but if we could show that the incomparability graph of $(3+1)$-free posets generated by permutations are $e$-positive we would have even more support for the conjecture. We explored this avenue briefly, but have not made progress with it.

Special thanks to our advisor Eric Egge for his guidance in this project.

## A Generating $(3+1)$-free posets

In looking at this topic, it is useful to have families of $(3+1)$ free posets that we can refer to. We looked at two methods in particular: generating posets using permutations and using unit intervals.

To generate posets using permutations, we let $P = \{(1, 2, \ldots, n) = \pi\}$. These posets are called order dimension 2 posets, because they are the intersection of 2 total orderings. We can then generate $(3+1)$-free posets by avoiding the subsequences $dabc$ and $bcda$ for $a < b < c < d$ (see [1] for further discussion).

**Example A.1.** Let $P = \{1, 2, 3, 4, 5\}$ and let $\pi = 45123$. This poset is not $(3+1)$-free because it contains the 3 element chain $1 \sim_\pi 2 \sim_\pi 3$ and $4 \not\sim_\pi 1, 4 \not\sim_\pi 2, 4 \not\sim_\pi 3$.

This method is useful because the posets generated from it are exactly the posets whose incomparability graphs are the comparability graphs for another poset. This means if $P$ is an order dimension 2 poset, then $\exists Q$ s.t. $inc(P)$ is the comparability graph for $Q$. A corollary of this is that $Q$ is also an order dimension 2 poset, so it is generated by a permutation. In fact, if $P$ is generated by $\pi$, $Q$ is generated by the reverse of $\pi$. We looked at order dimension 2 posets because they are simple and provide some structure that could make solving the poset chain conjecture easier for this special case.

The second family of $(3+1)$-free posets we looked at were unit interval posets. Let $P = \{(a_1, a_1 + 1), (a_2, a_2 + 1), \ldots, (a_n, a_n + 1)\}$, $\sim_{ui}$. We say that $(a_i, a_i + 1) \sim_{ui} (a_j, a_j + 1)$ if $a_i + 1 \leq a_j$. Unit interval posets are both $(3+1)$ and $(2+2)$ free. Using unit interval posets can give a reasonable application of graph coloring.

**Example A.2.** Suppose we have $P = \{(a_1, a_1 + 1), (a_2, a_2 + 1), \ldots, (a_n, a_n + 1)\}$ as defined above, and we consider the elements of $P$ as unit interval jobs. When we color $inc(P)$, no elements of the same color overlap. So we can view this coloring as an assignment of jobs to workers, such that no one has to work...
more than one job at a time. The number of colorings using \( n \) colors is the number of ways to assign the jobs to \( n \) workers.

## B The Jacobi-Trudi Identities

The Jacobi-Trudi identities give us some good ways of changing bases in the space of symmetric functions. Egecioglu and Remmel [3] and Gasharov [5] use these identities to relate complete homogeneous symmetric functions to Schur functions in their proofs of combinatorial facts. To make our paper more self-contained, we will give a brief overview of the Jacobi-Trudi identities and their use in the various parts of our paper.

The Jacobi-Trudi identities state that, for any partition \( \lambda \) and any \( k \geq l(\lambda) \):

1. \( s_\lambda = \det(h_{\lambda_i+j-i})_{1 \leq i,j \leq k} \)
2. \( s_\lambda = \det(e_{\lambda'_i+j-i})_{1 \leq i,j \leq k} \)

In our paper we only use the first Jacobi-Trudi identity.

In the proof of the first Jacobi-Trudi Identity, we note that the terms of \( h_n \) correspond to lattice paths beginning at height 1 with only eastern and northern steps, and that have exactly \( n \) eastern steps. We can construct this bijection by examining the \( h \)-weight of each path, which is the product of the heights at each eastern step. So the term \( x_1^2x_2 \) in \( h_3 \) corresponds to two eastern steps, a northern step, an eastern step, and then infinitely many northern steps.

Additionally, for any \( n \times n \) matrix \( A \) we can write:

\[
\det A = \sum_{\pi \in S_n} \text{sgn}(\pi) \prod_{j=1}^{n} A_{\pi(j),j}
\]

So if our matrix is the matrix \( H \), given by \( H_{i,j} = h_{\lambda_i+j-i} \), we can be clever and get a determinant consisting of a signed sum of lattice paths corresponding to terms in some complete homogeneous symmetric function. A sign reversing involution allows us to cancel out all such paths that cross any other path, so we end up with a sum of non-overlapping lattice path weights as our determinant, which we can easily put in bijection with semistandard tableau, completing the proof. See [4] for a more thorough treatment of this proof.

## References


