Necessary and Sufficient Conditions for Learning

A Journey in Teaching Calculus

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This book would not be possible
without the guidance and support provided by

Eric Egge
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Breakthrough Saint Paul

or without the courage, commitment, and compassion of

the 2009 Math 101 students

Teaching 101: One Philosophical Medley

My Background in Teaching

This book is a result of a term spent leading Tuesday and Thursday sessions of the Fall 2009 MATH-101 course: Calculus with Problem Solving. While Professor Eric Egge led the course 3 days a week (the Calculus part of the course), I was in charge of leading the two other days (the Problem Solving part of the course). So, while Eric introduced material, assigned homework and gave tests, I played more of a review and practicing role. This book contains all of the materials I created for the course, including lesson plans, worksheets, activities and review exercises (From the Stewart Calculus Textbook, Sixth Edition).

The purpose of this project, for me, was not to learn how to teach, but to learn how to teach *better*, part of the constant struggle for teachers everywhere. It is my firm belief that no matter how well you can teach, you can always learn better techniques. Furthermore, even the best techniques won't work for every class for every student, and in order to be an effective teacher you need to constantly be searching for new solutions to your problems.

While Carleton has provided me with a firm theoretical background in teaching, I do not think I would be nearly as prepared to undertake this project were it not for my practical background teaching for 4 summers at a teaching internship, Breakthrough Saint Paul. After learning various state-of-the-art teaching techniques and getting the opportunities to work with more than a dozen mentors, I have been able to piece together a medley of principles, techniques and tips that guide my craft.

That said, I must offer a disclaimer to anyone reading this book: While this philosophy works well enough for myself, every teacher must develop their own methodologies and accumulate their own way of teaching in their own unique style. We all have different influences and approaches to the issues of education, and while this diversity benefits the community at large, it can sometimes be a barrier to sharing teaching knowledge if we forget about this principle.

What I hope to offer with this introduction is a brief insight into my personal philosophy of teaching before I started teaching this class, so the reader will be better equipped to review these lessons and reflections and make sense of them. In the sections that follow I elaborate on my former views on various aspects of planning and implementing lessons. I have also tried to provide some research on relevant topics where appropriate. Finally, at the end of this book I will reflect on this framework and explain how it has changed. All I ask of you, the reader, is that you take these views with a proverbial grain of salt, and acknowledge the multiple ways that teachers can succeed in helping students achieve.

Active Participation

One of the things many teachers would admit to wanting a better solution for is how to engage their students with the material. We have all seen amazingly engaging speakers- but this is not what we mean when we say "engage with the material". In fact, you do not need to be outspoken or loud in order to really engage your students. The idea behind Active Participation comes from a desire to not only get a few students to answer questions or start a discussion in class, but to engage **every** student **every** day **every** 3-5 minutes. The benefits of having this structure in a lesson are two-fold. For one, instead of having one student at a time answer questions (and the teacher even answering some of them without giving students a chance to even think about the question), every student can participate and activate pathways in their brain. Also, providing such opportunities frequently in a lesson allows students time to

digest important information before jumping ahead, which actually decreases the amount of time needed to solidify the concept in the long run! Any technique that allows students to participate **frequently**, **consistently**, and **simultaneously** will produce these benefits; they are nothing too revolutionary, simply tools that fit under a revolutionary framework.

Active Participation does not necessarily mean having a prepared mini-activity every 3-5 minutes. Rather, it is a way to think about how to engage students more in every lesson without changing the whole structure of the lesson. It can be as simple as rephrasing a question such as "Who can tell me what 2+2 is?" into something like "Tell a partner what you think 2+2 is". The answer to the former may not be exactly what you are looking for as a teacher (the answer is either "I can", or "I can't"; the student doesn't immediately answer because they start to assess the perceived difficulty instead of working through the problem), whereas the latter question is processed immediately as something the student will try to do as long as they are able.

Rephrasing questions into directions offers one way for students to participate "covertly", that is to say that you don't always see them answering the problem but you can almost see the gears turning in their heads. For students who like to listen more or who are quieter, this form of participation may be more comfortable and beneficial. "Overt" participation is another way for students to participate, and leads to a whole new set of techniques to choose from, including working through a problem with a partner, drawing what they think a graph will look like, or making a sound that might describe the rate of change of a function (think Doppler effect). These are all "overt" because they are things that a teacher can observe and use to informally assess where students are in their understanding. The students who benefit more from this kind of participation are those who are more vocal and feel a strong desire to share all their ideas. The class will benefit as a whole from this kind of participation because these students will have an outlet for their ideas and may not feel the need to dominate whole-class discussions as much.

Many of these techniques are more age-specific than others, and in the conclusion of this book I will provide a list of techniques I have used and judge as appropriate for undergraduate classrooms. Again, the main point of Active Participation is to give plenty of opportunities for students to activate knowledge about the material in covert or overt ways.

Collaborative Learning

Within this framework, situations that involve more than just a pair of students fall under the category of collaborative learning, or groupwork. My approach to groupwork mostly comes from ideas outlined by Michaelson & Knight (1997) that relate to the factors involved in effective group assignments: individual accountability, discussion, direct comparison of performance outputs, and team performance. Their argument is that in order to justify creating a collaborative learning opportunity for students, something has to be gained that cannot be gained in an individual learning opportunity. When group assignments create cohesion in a group, these factors that are present in effective group assignments arise and the learning that occurs is exponential.

In order to create these opportunities, group assignments should focus on the 3 S's: Same Problem, Specific Choices, and Simultaneous Reporting. The authors argue that when students work on the **same problems** they benefit from collaboration because they are able to directly compare solutions "pound for pound". If students were to work on separate group assignments, they may as well be individual assignments. Similarly, making a list can be done individually, making a choice can be done individually, but making a **specific choice** and defending it reaps the most benefits from a collaborative setting. Finally, when students are able to better focus on questioning the choices of other groups instead of on defending their own choices (which is what **simultaneous reporting** allows) they learn more from the work of other groups.

Developing a Mathematical Identity

The final thing I want to comment on as a part of my framework going into this project relates to the "Mathematical Identity" of the student. Similar to other social identities, a person's mathematical identity consists of various factors including previous experiences, feedback from others, self-perception and self-esteem, and natural abilities. Whatever these students bring to the class, many are there because they have had negative experiences with mathematics. There are numerous studies that document the long-term effects of negative experiences with mathematics, and the results can be disturbing!

One reason students may have struggled is that their personal learning styles (related to Gardner's theory of Multiple Intelligences) may not be those favored in traditional math classes. Indeed, it seems natural that the Logical-Mathematical Intelligence dominates the realm of mathematics. However, for students who score low in this area on surveys, learning mathematics may be more difficult if done in a traditional way. Students who cannot easily visualize or intuit patterns may need to engage with the material in other ways, such as experimenting, drawing, talking or reflecting, rather than simply taking notes and working through problems.

Another reason students may have anxiety or difficulty performing in math classes is because of the intersection of social identities. Studies documenting false differences between students as a result of race or gender have highlighted different biases in curricula, classrooms and tests. The long-term effects of perceiving a difference where there is none under controlled circumstances are very negative and can lead students to second-guess themselves even when they have done everything correctly in a math problem.

As an educator, I believe it is my responsibility to first get to know students and their own unique mathematical identities and experiences, and work with students to overcome prejudices and misconceptions about mathematics. While Euclid may have said, "There is no royal road to Geometry", there certainly seem to be different roads that students can be on as they navigate math classes in today's education system.

In this spirit, on the first day of class I asked students to describe their mathematical experiences to me and take an online survey about multiple intelligences. Interestingly enough, these patterns that I have studied did come out in their responses– students thought they were horrible at math, or had other misconceptions about what is required to be "good" at math. Also, approximately two thirds (2/3) of the students scored their top three intelligences as Kinesthetic, Interpersonal and Intrapersonal, whereas only a handful of students had Logical-Mathematical, Spatial or Linguistic intelligences in their top three. As a result, I will attempt to cater to these intelligences as much as possible when reviewing or introducing concepts. The ultimate goal for this project is not to benefit my own teaching as much as it is to provide students with service. My hope is to positively impact these students and help them develop their own mathematical identities.

How To Use This Book

This is by no means a book of formulas for leading the Tuesday/Thursday sessions of this course. It is more like a record of how I did it within my own philosophical framework and abilities. Because anyone attempting to lead this course will most likely have a different framework and different (although not particularly better or worse) abilities, it may not benefit them to try to replicate what I have done lesson for lesson. Instead, it may be beneficial to pay attention to the structure and guiding principles used to construct the lessons.

How To Read These Lessons

On the following page is a sample lesson with boxes highlighting areas of interest:

Objectives & Assessments: Each objective is explicitly matched with a way to assess students on that objective that is built into the lesson. This way, when designing the body of the lesson, a teacher is able to intentionally build in opportunities to see how the lesson is going, and adjust as necessary. Students will not always tell you when they don't understand a given concept, so it is partially the responsibility of the teacher to informally assess where students are.

Vocabulary: Vocabulary in mathematics? While most of the students in the class consider math to be more about formulas and calculations, there are a variety of vocabulary-related obstacles that students need to tackle before they can understand concepts. For undergraduate majors in mathematics, they are used to rigorous definitions, and even when a word has other meanings in the vernacular (rational, limit, integral, acceleration, etc.) they are expected to learn a new, context-specific definition. The students in this class may not be used to this, and as the teacher I always want to be thinking about how I introduce, practice, and use the essential vocabulary throughout a lesson.

<u>Lesson Body</u>

Level of Interactions: There is a spectrum of interaction with material that goes from teacher-centered (think lecture-style where the material is introduced by the teacher, putting all the responsibility of learning on the teacher, leaving the student few ways to learn by themselves) to student-centered (times when students need to make sense of material by themselves, putting the responsibility for learning on them and allowing the teacher to be more of a guide rather than broker of knowledge). In my framework, there are benefits to both ends of the spectrum, although my own bias is to try to do more student-centered lessons because there are a lot of teacher-centered classes out there already, and it is easier to cater to Interpersonal and Kinesthetic intelligences through a student-centered approach. This approach is similar to a constructivist-based one, although leaves less open to interpretation (which is good for those students who just want you to *show* them the "right" way to do a problem). Ultimately I think that the best lessons are those that have a balance and those that have flexibility for the teacher to adjust if students need one approach more than another on a particular day.

Multiple Intelligences: Again, in an effort to cater to the class' overwhelming number of students whose intelligences are not suited to traditional learning styles of mathematics, I want to be conscious of what kinds of intelligences I favor in my lessons. As with the student-centered approach, I want to have some flexibility to appeal to multiple intelligences in a single lesson.

Functions – Essential Models of Functions September 15, 2009

Overview: This lesson presents the material of section 1.2 in the textbook on the various categories of functions. Students will use a constructivist method to classify various functions in different forms (algebraic, graphical, descriptive and table) in a small group. The activity is designed to activate thinking about the propertie thematical models. The concluding con though there are other ways to categ oduction to the idea of "mathemati em to reflect on their own mathema

Each objective is phrased as "Students Will Be Able To…" and each one is paired with an assessment to ensure that the teacher can measure progress toward the objective.

Objectives	Assessments
• Understand and use the procedure used to develop mathematical models	• Take a group of functions and find the common properties to develop a mathematical model
• Recognize and classify functions based on their properties	• Essential Models activity
• Compare essential models of functions	• Read other groups' posters and challenge their classifications.

Vocabulary:

Social Identity	Polynomial
Mathematical Identity	Power
Linear	Rational

The Vocabulary section just serves as a reminder to introduce essential vocabulary before using it in a lesson!

Materials/Preparation:
Sets of functions (7)
Moodle Website on Projector

Whatever the introduction and wrap-up are, it is important to consider the structure of the class so that students can more easily become aware of their metacognitive experiences and make connections to previous classes.

Introduction/Hook:
Take a few minutes to introduc
Comps project. Give some personal background, and let the students know that you are excited to have the opportunity to work with them!

Lesson Body	Time	Inter.	Styles
<u>Essential Models Activity</u> • Ask students to refer to the reading due for Wednesday, and ask someone w ort of ll be	5	M	Linguistic
• set of to choose the most appropriate classification for the functions. They can use their books if they do not remember from the reading.	10	SC	Inter. Logical Linguistic Spatial

The activity information will be in this area, and is a general set of instructions about the activity. More detailed worksheets and examples will follow each lesson.

The three columns to the left of the lesson body are for ***estimated time***, ***level of interactions*** (TC=Teacher-Centered, SC=Student Centered, M=Middle), and ***learning styles*** engaged by the activity (based on Gardner's 8 intelligences).

Wrap up:
On the projecto
Wednesday nig
them before. Sh
and see if they get them right. They may not, and this could be a good opportunity to illustrate that our intelligences may not always be noticeable.

Unit One – Functions, Limits and Derivatives

General Overview: The first week will be very important in figuring out where the students are in terms of their algebra skills. Devoting a whole week to reviewing functions and their graphical and algebraic representations will help ensure a strong base on which to build the knowledge of calculus. Also, it will be important for me personally to get a handle on how working with college students is different from working with middle school students. I have thought of many activities to explore the ideas in the course, and one of my concerns is that they will not be appropriate for this age group. I'm sure I will learn a lot in the first few days! I will have to intentionally set a tone of open dialogue so that students will feel comfortable challenging my methods constructively.

Major Objectives
• Recognize and classify functions based on their properties (algebraic and graphic) • Predict the shape of a function based on its algebraic properties and categorization • Find the domain of a function based on its algebraic properties and categorization • Understand the meaning of limit notation in reference to a graph • Predict when the limit of a given function will exist or not at a given point • Use the definition of a derivative to find the derivative of a given function at a given point • Apply the concept of a derivative as the limit of slopes of secant lines to the slope of a tangent line

Date	**Lesson**	**Page**
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09/17	Functions – Essential Models of Functions (cont.) • Algebraic Properties • Graphic Properties • Exponential Functions	13
09/22	Limits – Introduction to Limits • Questions about Limits • Limit Telephone	16
09/24	Limits – Derivative as a Limit of the Slope of Secant Lines • Unfamiliar Problems • Log Laws • Graph Pictionary	20

Functions – Essential Models of Functions September 15, 2009

Overview: This lesson presents the material of section 1.2 in the textbook on the various categories of functions. Students will use a constructivist method to classify various functions in different forms (algebraic, graphical, descriptive and table) in a small group. The activity is designed to activate thinking about the properties of functions and how mathematicians organize them based on mathematical models. The concluding comments emphasize the conventional nature of these categories even though there are other ways to categorize functions. Seeing as this is the first lesson, there is also an introduction to the idea of "mathematical identity" for the students, and an assignment related that asks them to reflect on their own mathematical identity.

Objectives	**Assessments**
• Understand and use the procedure used to develop mathematical models • Recognize and classify functions based on their properties • Compare essential models of functions	• Take a group of functions and find the common properties to develop a mathematical model • Essential Models activity • Read other groups' posters and challenge their classifications.

Vocabulary:

Social Identity	Polynomial	Algebraic
Mathematical Identity	Power	Trigonometric
Linear	Rational	Exponential

Materials/Preparation:
Sets of functions (7)
Moodle Website on Projector

Introduction/Hook:
Take a few minutes to introduce the Tuesday/Thursday component of the course and how it fits into the Comps project. Give some personal background, and let the students know that you are excited to have the opportunity to work with them!

Lesson Body	**Time**	**Inter.**	**Styles**
<u>Mathematical Identity</u> • Tell students that part of studying mathematics is engaging in discussion with other students of mathematics, and it helps to get to know each other first. See if anyone can define the term "social identity", and then ask students to think of examples of social identities as a brainstorm.	7	M	Intra.
• After getting 7-10 up on the board, introduce the idea of a mathematical identity, and discuss how these identities might intersect and what that means for this course.	8	M	Visual Inter.
<u>Essential Models Activity</u> • Ask students to refer to the reading due for Wednesday, and ask someone who has read to give a basic overview of the section. Tell students they will sort of be using the procedure for developing a mathematical model, only they will be trying to figure out what classification several functions fit into.	5	M	Linguistic
• Split the students up into groups by numbering off, and hand each group a set of functions in different forms. Tell students they will have about 10 minutes to choose the most appropriate classification for the functions. They can use their books if they do not remember from the reading.	10	SC	Inter. Logical Linguistic Spatial

<u>Essential Models Questions</u> • Post the papers from each group along the walls in the room, and tell students they will have 10 minutes to go around and read the other groups' work and write down questions or comments about the classifications. Their task here is to provide constructive criticism and to challenge the assumptions that other groups' have made.	10	SC	Intra. Linguistic Logical
• Provide some time for students to discuss their questions and comments by addressing each other directly. Jump in when clarification on the conventions is necessary, but otherwise students should discuss these things with each other.	10	SC	Inter. Logical Linguistic
• Comment on the conventions that mathematicians have used to set up these models and discuss the procedures that the students used today to classify functions.	5	TC/M	Linguistic

Wrap up:
On the projector, show students how to access the moodle website and direct them to the assignment for Wednesday night. Introduce the idea of multiple intelligences briefly for students who have not heard of them before. Share your own top three intelligences, having students guess what they think they might be, and see if they get them right! They may not, and this could be a good opportunity to illustrate that our intelligences may not always be noticeable. Address the questions that they have to answer on the homework and see if there are any questions about what they need to write. Otherwise remind them about office hours and see if Cassat will work as a location.

Homework:

List the last three math classes you have taken.

Describe some of your positive and negative experiences with math.

List some things you enjoy about math and list some things you do not enjoy about math.

Take this Multiple Intelligences Assessment and list your top three Intelligences. Do you think these are accurate? <http://literacyworks.org/mi/assessment/findyourstrengths.html>

All of these questions get at your unique "mathematical identity". Briefly share what you are comfortable sharing about how your mathematical identity intersects with other social identities, and what this may mean for you in this course. How have other people's perceptions of your identity shaped your mathematical experiences, either positively or negatively?

Reflection on Lesson September 16, 2009
(Functions – Essential Models of Functions September 15, 2009)

General Reflections:
Based on conversations with students who visited my office hours, the message about mathematical identity came across. Many students were surprised by the concept, but a few had obviously already thought about it before. Also, after working with students only once or twice, I am already starting to see that some struggle with math a lot more, especially with algebra and arithmetic.

Were the objectives met?
• Understand and use the procedure used to develop mathematical models
I'm not completely sure if students understood this concept. The models were so confusing to them that it wasn't clear that mathematicians and scientists had developed them as general categories, rather than innate attributes of the functions.

• Recognize and classify functions based on their properties
This was very difficult for students, especially the less familiar categories (rational, algebraic). One of the ways that it was difficult was the substantial overlap between some categories (power functions fit into most of them!)

• Compare essential models of functions
Those students who were able to categorize functions were able to start to appreciate the differences between categories (more so graphically than algebraically).

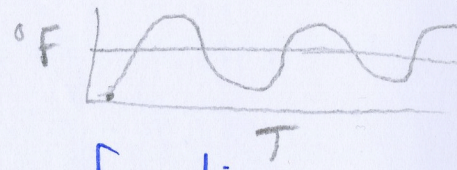
Were the assessments effective?
• Take a group of functions and find the common properties to develop a mathematical model
Did not end up doing this– most students had not read through the section and so I just introduced the categories with examples of each.

• Essential Models activity
The students seemed adequately challenged by the functions, but they simply didn't have enough background to make informed decisions about classification. They correctly identified many characteristics of the functions but were not ready to classify them.

• Read other groups' posters and challenge their classifications.
The questions that students came up with for each other were appropriate considering that they were not ready to sort the functions. The questions were the part that really helped them the most in clarifying some of the questions they had.

Ideas for Improvement
Overall I think the students learned more about these functions than they knew before, but were still very confused about the organization of the categories. I think in doing this next time I would have more explicitly gone over the categories and used this activity as practice rather than review. Students seemed very confused about the categories that overlapped, and perhaps doing an example of a function that fit into many of the categories before starting the activity would have opened their minds to classifying the functions. Many of the students chose the first categories that came to mind, even if one of the four functions did not fit into that category. Also, there was not enough time to address all the questions, so this activity might be better done in one whole class period.

Group # 2

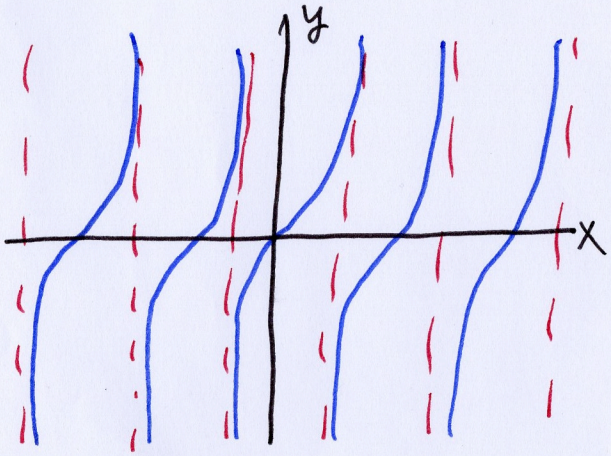


(a) average air temperature T as a function of the month m over a span of 5 years.

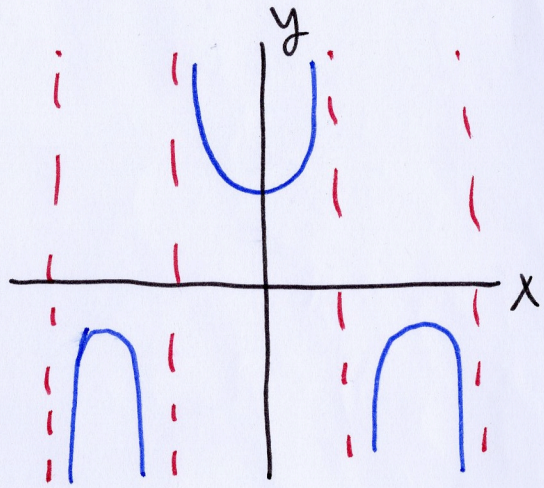
(b)

x	-3	-2	-1	0	1	2	3
f(x)	-.99	-.42	.54	1	.54	-.42	-.99

(c)



(d)



Choose the most appropriate classification for your set of functions. List all the reasons you can come up with to explain your classification. Be specific!

(a) sin (x) → trig
- starts at min. and grows to max, forms wave

(b) cos (x) → trig
- at zero it is at a max and then goes to a min. on both sides

Nice descriptions!

(c) tan(x) → trig
- the x^3 function repeated over and over w/ asytopes

(d) sec (x) → trig
- inverse of cos(x) so it is center over x-axis

Group # 4

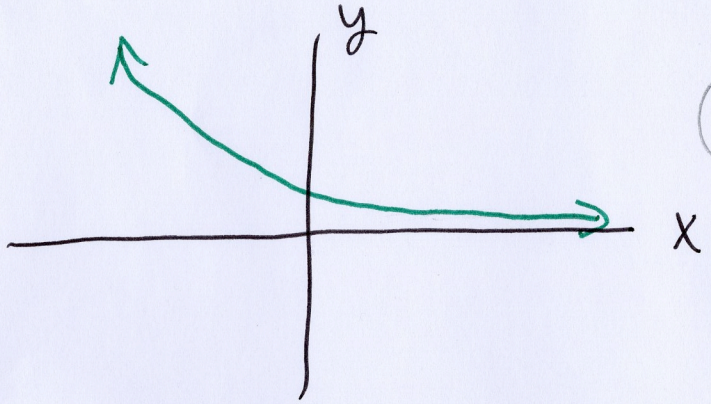
(a) $f(x) = \left(\frac{1}{2}\right)^x$

(b) population P in a country after t years

(c)

x	-3	-2	-1	0	1	2	3
f(x)	1	1	1	1	1	1	1

(d)



Exponential

Nice!

Choose the most appropriate classification for your set of functions. List all the reasons you can come up with to explain your classification. Be specific!

a) exponential, $\frac{1}{2}$ is a pos. constant and to power of x

b) polynomial, linear if the growth / decline is constant, exponential normally used to graph natural phenomena

c) ~~either it's constant~~ Exponential, shows data close to the asmyptote

d) Exponential or polynomial, it decreases at a decreasing rate

Functions – Essential Models of Functions September 17, 2009

Overview: The rest of chapter 1 explores the various mathematical models presented in the first two sections, so today we will be reviewing, clarifying the various models and focusing on their graphic and algebraic properties. As students approach problems with unfamiliar functions they will need to be able to know basic properties of these functions based on their classification. This is especially important for visualizing tangent lines when derivatives come up later. Also, when looking at limits and domains it will be important to recognize how to manipulate rational and algebraic functions.

Objectives	**Assessments**
• Predict the shape of a function based on its algebraic properties and categorization. • Find the domain of a function based on its algebraic properties and categorization. • Explain the relationship between exponential functions, growth and decay.	• Solicited answers to example problems (especially power functions) • Solicited answers to example problems (especially rational and algebraic functions) • Solicited answers to example problems

Vocabulary:

Linear	Power	Algebraic	Exponential
Polynomial	Rational	Trigonometric	

Materials/Preparation:
Examples of power, rational, algebraic and exponential functions

Introduction/Hook:
Remind students that the classifications learned on Tuesday are difficult to understand, and that today they will be reviewing them as well as looking at their algebraic form and their graphs.

Lesson Body	**Time**	**Inter.**	**Styles**
<u>Algebraic Properties</u> • Make an outline on the board of all the categories (except trigonometric and exponential) and their algebraic properties by asking guiding questions and soliciting answers from what students already know. What are their parameters? What kinds of functions fit under multiple categories? Which is the category they all fit into? As much as possible get examples up on the board by asking students to list functions they know.	15	M	Logical Linguistic Intra.
• Briefly discuss the domain of each function, spending time discussing why rational and algebraic functions might have peculiar domains.	5	M	Logical
<u>Graphic Properties</u> • Examine the parameters of the functions further by showing how the graphs of each functions change along with the parameter. For example, look at the odd and even degrees of power functions. For rational functions, look at their "pseudo-degree" to see an overall shape. Solicit students for the shapes that they already know. As much as possible ask students to predict the changes that occur when the parameter is changed so they begin to see the relationships.	15	M	Logical Spatial Intra.
<u>Exponential Functions</u> • Begin the discussion about exponential functions by clarifying the difference between exponential functions and power functions. Discuss what happens with the base parameter as it changes. Can it be negative? What if it is less than 1?	10	M	Linguistic Intra.

• Show the graphic properties of exponential functions, including growth, decay and comparison to power functions by looking at a few examples. In each case have students make predictions about how each graph will look and why.	5	M	Logical Spatial Intra.

Wrap up:
Ask what kinds of questions students have about the algebraic and graphic properties of the functions covered. Otherwise spend the remaining time addressing homework concerns.

Reflection on Lesson September 18, 2009
(Functions – Essential Models of Functions September 17, 2009)

General Reflections:
Students seemed more relieved after today because of the clarifications made with the different categories. Most of them seemed to understand the relationships between parameters and the shapes of graphs, although they are still learning some of these relationships.

Were the objectives met?
• Predict the shape of a function based on its algebraic properties and categorization.
Students struggled the most with the root and power functions, especially the even root functions. Based on conversations during office hours, they still do not fully understand the concept of root functions other than the square root, both algebraically and graphically.

• Find the domain of a function based on its algebraic properties and categorization.
Again, students are still struggling with this concept, and the connections between graphic and algebraic reasoning are still weak.

• Explain the relationship between exponential functions, growth and decay.
Most students were familiar with these terms and seemed to recognize the properties of the base parameter.

Were the assessments effective?
• Solicited answers to example problems
This method was effective in seeing where students currently were in their knowledge and reasoning related to mathematical models. While the whole-group discussion with a teacher-centered focus of clarifying definitions may have caused some students to remain quiet, the questioning and critical atmosphere helped students get to their misunderstandings. Several students asked very specific questions that showed they were starting to grasp the concepts by comparing and contrasting functions.

Ideas for Improvement
A handout may have been good for future reference, one that has connections or a diagram showing the relationships between kinds of functions. Although it was a good lesson on the complicatedness of the categories, it might be nice to have a clean copy. The structure and teacher-centered aspects of the lesson helped students feel more comfortable with the material than the student-centered aspects from the previous lesson. Perhaps if these two lessons were switched students would experience more success with the categorization activity.

Limits – Introduction to Limits September 22, 2009

Overview: Chapter 2 introduces the concept of a limit. Some of the research suggests that using the proper language of limits ("arbitrarily close", "approaches", "intends") can help students start to understand the somewhat troublesome concept of a limit. In fact, many students will benefit from exploring what a limit actually is before they can really understand the definition. I prepared an activity that will help students see what aspects of limits they need to pay attention to, graphically.

Objectives	Assessments
• Understand the meaning of limit notation in reference to a graph • Predict when the limit of a given function will exist or not at a given point • Translate between graphic and symbolic notations of limits	• Discussion after videos • Sample problems in discussion • Limit Telephone

Materials/Preparation:
Cue the video at (www.calculus-help.com/funstuff/tutorials/limits/limit01.html), prepare the half sheets of paper for Limit Telephone (writing 1-3 pieces of information about a graph using limit notation)

Introduction/Hook: (15 minutes)
As a review of what they were introduced to the day before, as well as a new way to think about limits, show them a few of the videos (pre-selected), pausing to make comments tying these new concepts together with ones learned the week before about the shapes of graphs. Really emphasize the point that limits involve approximations and that the value of the function at the point in question is not important.

Lesson Body	Time	Inter.	Styles
Questions about Limits • Open up a discussion for students to share their questions about limits. If they do not have questions to address immediately, put up a few examples of some limits and see if they can evaluate them without trouble. It will be valuable to let them discuss a little bit without giving away answers, but eventually telling them the correct process will help them understand what a limit is. Sample questions or prompts are included.	20	M	Logical Linguistic Intra. Inter.
Limit Telephone • Split the class up into groups of no more than 6. For each person in a group, prepare a distinct half-sheet of paper (sample included) with a prompt at the top, consisting of a written description of a graph, using only information in limit notation. The students' task will be to draw an example of that graph below, fold over the top of the sheet of paper, and pass it to the next person (clockwise). The next person should only be able to see the graph, and their task is to write information about the graph using limit notation, fold over the graph and pass it on. The activity should alternate in this way until the papers have been to everyone in the group. The group should then analyze their papers to see if they preserved the essential information given in the prompt. What information was left out? What (extraneous) information was added? • As a large group, briefly discuss the implications of the activity, and make a verbal list of the essential components of a limit.	20	M	Logical Spatial Intra.

Wrap up:
If there is any time at the end, address homework questions.

Reflection on Lesson September 23, 2009
(Limits – Introduction to Limits September 22, 2009)

General Reflections:
The videos seemed to help many students visualize the concept of limits in a different way. They asked many great questions that showed they were really engaging with the idea of approximation. There was a fairly wide gap in the class between students who seemed to get the concept right away and those that still were struggling with it, rendering some of the activities difficult to manage. For the most part students were understanding of this challenge, but some got frustrated because of already understanding limits or not understanding them at all.

Were the objectives met?
• Understand the meaning of limit notation in reference to a graph
Most students seemed to understand the idea of a limit, but still struggle with the graphical interpretation. Some of them did not recall (or had never learned) symbols such as a filled or open circle on the graph. Additionally, some of them did not truly understand the idea behind a "hole" in the graph at a single point.

• Predict when the limit of a given function will exist or not at a given point
This is the part that many students struggled with, but by the end of the class they seemed to be in a better place as far as having an intuition about the limit at a point with a given set of circumstances.

• Translate between graphic and symbolic notations of limits
This was still rough, but most students were able to focus on the important aspects of limits: the two-sided limit instead of the value of the function (if it is even defined).

Were the assessments effective?
• Discussion after videos
There was not much discussion during the videos, but the students seemed to be able to answer the questions I asked and understand the points made in the videos.

• Sample problems in discussion
These stimulated a little more discussion, although some of the problems generated more meaningful discussion than others. Overall, though, students did not seem too bored with the questions, even if they answered them easily.

• Limit Telephone
The students were very confused at first, but I think this confusion resulted because they were still engaging with the ideas at hand. By phrasing their questions and getting them answered by me and their peers, they eventually learned to start ignoring irrelevant info (value of the function or undefined value at the point in question) and focus on the different possible ways a limit can or cannot exist at a point (one-sided and two-sided limits). Most of the pictures turned out "correct" at the end!

Ideas for Improvement
I think one way to improve the Limit Telephone activity (some students got a little too frustrated) would be to do an example together, leaving the activity to solidify the correct understanding of limits rather than reinforcing incorrect ideas.

- Suppose that a function f(x) is defined for all values except x=2. Can anything be said about the existence of the limit of the function at x=2?
- Suppose that a function f(x) is defined for all values in [-1,1]. Can anything be said about the existence of the limit of the function at x=0?
- If f(1)=5, must the limit at x=1 exist? If so, does it equal 5? Can we conclude *anything* about the limit at x=1?

When finding the limit at a given point, does the value of the function at the given point even matter?

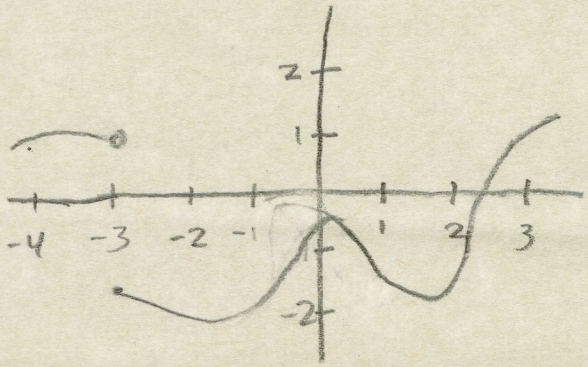
- If the limit at x=1 of a function equals 5, must the function be defined at x=1? If it is, must f(1)=5? Can we conclude *anything* about the function at x=1?
- What does it mean that "a function *intends* a certain height"?
- How could a function fail to intend a certain height?

How can you know for sure if a function has a limit at any given point?

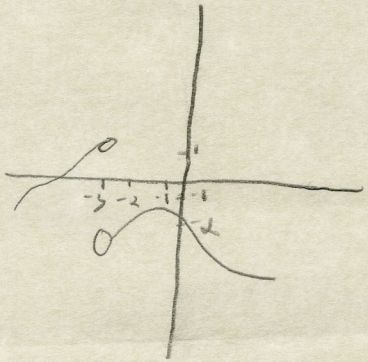
- When will substitution work?
- When will substitution fail?
- What could you do if substitution fails?

How do you know that algebraic manipulation will work to find a limit?

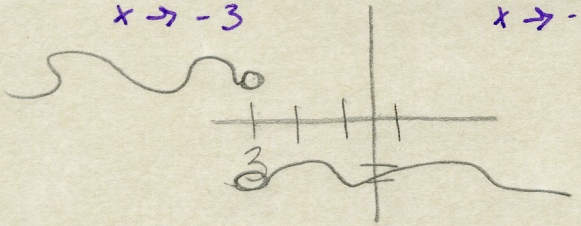
$\lim_{x \to -3} f(x)$ d.n.e. (does not exist)



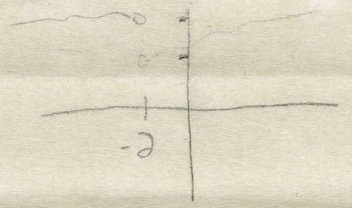
$\lim_{x \to -3^-} f(x) = 1$ $\lim_{x \to -3^+} f(x) = -2$



$\lim_{x \to -3^-} f(x) = +1$ $\lim_{x \to -3^+} f(x) = -2$

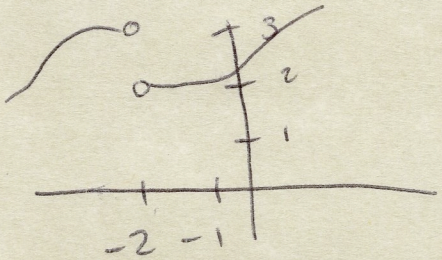


$\lim_{x \to -2^+} f(x) \neq \lim_{x \to -2^-} f(x)$



$\lim_{x \to -2^-} = 3$ $\lim_{x \to -2^+} = 2$

$\lim_{x \to -2^-} = 3$ $\lim_{x \to -2^+} = 2$



Limits – Derivative as a Limit of the Slope of Secant Lines September 24, 2009

Overview: We skipped section 2.6 and went straight to setting up the concept of a derivative by looking at the limit involved with secant lines. With a test coming up on Monday, the focus of today's lesson was on strengthening this concept as well as reviewing the material covered for the test. To help ease some of the anxiety about the first exam, I also gave them some guidelines for handling "unfamiliar problems" that they might encounter on the exam.

Objectives	Assessments
• Use guidelines to solve "unfamiliar problems" that they have not solved before	• Sample problems as a class
• Use the definition of a derivative to find the derivative of a given function at a given point	• One of the sample problems
• Apply the concept of a derivative as the limit of slopes of secant lines to the slope of a tangent line	• One of the sample problems

Materials/Preparation:
Unfamiliar problems PowerPoint

Introduction/Hook:
Give an overview of the class, letting students know we will be focusing on reviewing for the test.

Lesson Body	Time	Inter.	Styles
<u>Unfamiliar Problems</u> • Briefly discuss why having some guidelines to follow for handling unfamiliar problems is beneficial! Introduce the guidelines in the PowerPoint, and follow with a few examples done together of how the steps work. Make sure to cover derivatives in these examples since they haven't had as much exposure to that material. (see Guidelines and sample problems)	20	M	Logical Intra.
<u>Log Laws</u> • Ask students to shout out the log laws that they remember. Once they are all on the board, do a little review by having students see a few "wrong" examples of log laws, and challenge them to correct them or point out the flaws (See sample problems).	5	SC	Logical Intra. Inter.
<u>Graph Pictionary</u> • As a simple review of the shapes of functions, split the class into 3 teams. They will play Graph Pictionary, in which one person from each team comes to the board and will be given a slip of paper with a function in algebraic form. Their task is to plot the function on the board (they can put up axes before starting) and their team must guess the algebraic form of the function. The first team to get their function gets a point. The person can label points or draw asymptotes to help out.	10	SC	Inter. Linguistic Logical Spatial

Wrap up:
Address any homework concerns the students may have. Also, let them know about the review problems from their book so that they can use those to study. Remind them about the exam review session on Sunday night!

Reflection on Lesson September 25, 2009
(Limits – Derivative as a Limit of the Slope of Secant Lines September 24, 2009)

General Reflections:
I think the students are ready for the exam, for the most part. They are anxious, but they have a good understanding of most of the concepts that were review, and those few students who do not have a good handle on these were able to understand limits and the concept of derivatives fairly well. After only two weeks, it seems very soon to have an exam, but they seemed to receive the guidelines for dealing with unfamiliar problems well. Their biggest concern was not knowing how Eric's exam would be since they had never had one before, so I attempted to put some "worst-case scenario" problems out there for them to grapple with, all the time being transparent that there would most likely be only a few very difficult problems on the exam.

Were the objectives met?
• Use guidelines to solve "unfamiliar problems" that they have not solved before
Students were able to follow the guidelines along with me– that said, this gives no indication of how they might fare alone. Hopefully they took away something from these guidelines, at the very least a sense that they should not panic if they don't know how to solve a problem immediately!

• Use the definition of a derivative to find the derivative of a given function at a given point
Students know how to set up the limit, but evaluating it still gives some of them difficulty. Those students with poor algebra skills have the hardest time, whereas students who do not immediately see how they might algebraically manipulate the fractions get easily frustrated. Doing a wide variety of these problems will be good for them!

• Apply the concept of a derivative as the limit of slopes of secant lines to the slope of a tangent line
This was newer for them, as they have not yet solidified the idea that the derivative represents the slope of a tangent line. Those students still confused by the limit calculation especially had trouble going beyond that to apply the concept to a graph.

Were the assessments effective?
• Sample Problems
The problems were good for everyone because they were new, or unfamiliar, but definitely within their ability to solve. Those who felt confident were challenged by the different format of the problems, and those that were not as confident were able to understand the solution we found as a class. I wish we could have done 10 of these problems, but there wasn't very much time! I did suggest to them that looking in the book for problems that they could not immediately solve and grappling with them for a while would be the best way to study for these problems.

Ideas for Improvement
Perhaps we could have covered more problems if the students had worked in groups instead of reviewing HW. Part of the issue of reviewing HW problems is that not everyone benefits from seeing them at board if they already did them or knew how. Using this time to do similar problems that are new for everyone might be better.

Math 101 Exam One September 28, 2009

1. In this problem let $f(x) = x^2 + 3x$, let $g(x) = 3^x$, and let $h(x) = \sin(x)$. Find each of the following.

   (a) $f(2a)$

   (b) $f(3 + a)$

   (c) $(g \circ f)(x)$

   (d) $(f \circ h)(x)$

2. (a) Write the definition of the derivative of $f(x)$ at $x = a$.

   (b) Explain in a few sentences how the definition you gave in part (a) actually computes the slope of the tangent line to $y = f(x)$ at $x = a$. (Don't tell me how to plug $a + h$ into f. I want to *understand* why the definition you gave actually finds the slope of the tangent line.)

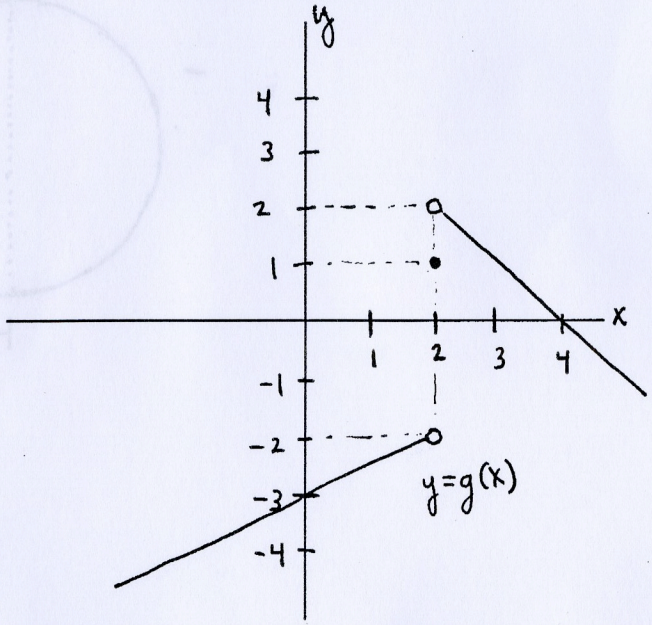
   (c) Use the definition you gave in part (a) to find $f'(3)$ where $f(x) = x^2 - 4x$.

3. (a) Sketch the graph of $y = 2^x$ on the axes below. Be sure to indicate the scale on your graph, and label at least two points on your graph with their coordinates.

   (b) How can we obtain the graph of $y = 3 \cdot 2^{x+4}$ from the graph of $y = 2^x$?

   (c) Sketch the graph of $y = 3 \cdot 2^{x+4}$ on the axes below. Be sure to indicate the scale on your graph, and label at least two points on your graph with their coordinates.

4. In this problem $g(x)$ is given in the graph below.



For each of the following, find the quantity or explain why it does not exist.

   (a) $g(2)$

   (b) $\lim_{x \to 2^+} g(x)$

   (c) $\lim_{x \to 2^-} g(x)$

(d) $\lim_{x \to 2} g(x)$

(e) $\lim_{x \to 3^+} g(x)$

(f) $\lim_{x \to 3^-} g(x)$

(g) $\lim_{x \to 3} g(x)$

5. Find $f(x)$, $g(x)$, and $h(x)$ so that $(f \circ g \circ h)(x) = \cos^4(\sqrt{x-4})$.

6. Find each of the following limits.

(a) $\lim_{x \to 3} \dfrac{x^2 - 5x + 6}{x - 3}$

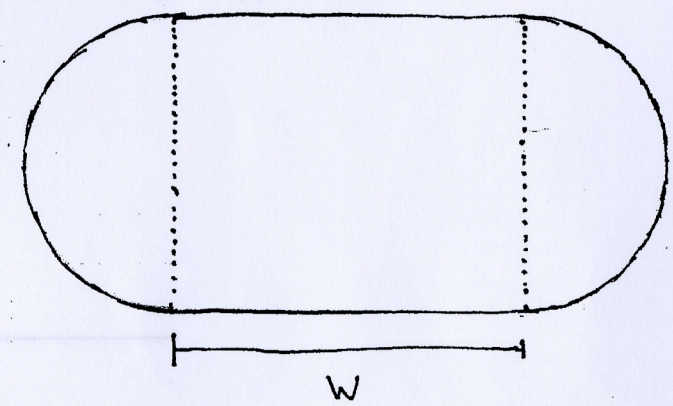
(b) $\lim_{x \to 2} \dfrac{\sqrt{x+7} - 3}{x - 2}$

7. Compute each of the following exactly.

(a) $\ln(e^5)$

(b) $e^{-3 \ln 2}$

8. A certain track consists of a rectangle with two semicircles at opposite ends, as in the picture below. If the perimeter of the track must be 400 meters, find the area of the field enclosed by the track as a function of the width w of the rectangle.



Exam One Review Problems (Chapters 1, 2)

Instructions: There are too many problems to do them all! This is a list of problems that will help you practice both familiar and unfamiliar problems. I suggest that you look at **all** of the problems and **attempt** the ones you don't immediately see how to do. During the review session, bring questions about the problems you do not know how to do.

Steps for solving "unfamiliar" problems:

1) What does the problem want me to find?
    a. What will my answer look like?
    b. What do I want to have appear or disappear?
2) Make a mental (or written) list of what you know about the problem:
    a. What approaches will or will not work?
    b. What steps do I know I will have to take?
    c. What problem does this one look similar to, even slightly?
3) What can I figure out using what I know?
    a. What would happen if I tried this approach…?
    b. Can I get part of the way, and then reevaluate?

Problems (note the two Review Problems R1 and R2, created by Daniel):

Section 1.2: 1, 2, 19, 20 (These are simple identification)

Section 1.3: Anything from 9-24 (To practice making horizontal and vertical changes)

28, 37-40, 47

R1: Suppose a parabola has a vertex at (3, -1) and another point at (5, -5). Using this information, find the equation of the parabola by figuring out how the graph x^2 had been shifted.

R2: Let f(x) = 2 ln (x-3) + 4.

(a) Find the equation of g(x) if the graph of g(x) is just the graph of f(x) that has been shifted down 6, shifted right 1, and compressed by a factor of 3.
(b) Find the equation of h(x) if the graph of h(x) is just the graph of f(x) that has been compressed by a factor of 3, shifted down 6, and shifted right 1. Is h(x) different from g(x)? In other words, does the order of these operations matter?

Section 1.5: 8-12 (If you need practice working with the shape of exponential graphs), 18

Section 1.6: 35, 36, 37-39

Section 2.2: 13-16 (and explain which limits exist at different points and why)

Section 2.3: Any of 11-30 are good practice!

Section 2.7: 6, 8, 19, 21, 22, 32-36

Concluding Comments

Unit One – Functions, Limits and Derivatives

Effectiveness of Methods:

• Student-Centered Activities: In my efforts to provide ways for students to engage directly with the material, I may have frustrated students who felt uncomfortable with the material. This may be attributed to a lack of experience with learning math this way or a general lack of self-confidence with mathematics. However, I concluded in a couple lessons that I would have liked to have more review of the new material and use these activities to practice rather than continue to introduce. The students would probably have had more success with the activities if they had a little more experience with the material, especially the Limit Telephone activity. While I still think a student-centered approach makes sense for many reasons, I will have to try to come up with some ideas for how to make it more appropriate for my students.

• Discussions & Group Work: In a few lessons I attempted to spur some discussion about the concepts they were learning, specifically with Limits. In my limited experience, I prefer to do this in smaller groups so more students have a chance to interact. However, the wide range of ability and confidence within groups is proving difficult to manage. Some students in randomly assigned groups reach conclusions quickly and the rest of the group will mimic their answers without really understanding the solutions. I will also have to revisit the structure of the group work I do in class to see if I can manage this issue.

• Sample Problems: Picking problems for students to work on has been one of the most time-consuming parts of preparation, yet I feel like I need to spend even more time on it! In several instances I have chosen problems without working them out the whole way, and it can be embarrassing to run into a mistake in the middle of class. I have tried to be transparent with my students about making mistakes, but it feels like some of them are starting to lose confidence a little. Additionally, it has been difficult to choose problems that appropriately match what they do on the other days of the week, and when I do not visit Eric's class I sometimes choose problems that are too difficult or too new for the students.

Ideas for Improvement:

As we move into even more new material, I will have to be conscious about how the concepts are presented by Eric, and later by myself, so that they match well. It will be even more important to have sample problems and let students practice, practice, practice! I still want to try to incorporate group work and student-centered activities, though, so I will do some research into the structure of these kinds of activities. I already know of one resource that may be able to give me some thought about the group work on problems, but I will also ask some professors in the department if they have some thoughts, because often times conferring with colleagues can be the best way to get new ideas!

Unit Two – Finding Derivatives

General Overview: As the course moves into more and more new material, my approach has changed to try to make the lessons more example-heavy. Now that we have explored the concept of a derivative, there is a lot to learn about how to evaluate and apply derivatives. Because of the intense amount of algebra involved, students will be moving at vastly different paces through the material. In an attempt to utilize group work to tackle example problems, I have tried to split up some of the tasks within groups. First of all, I will try to make a variety of problems at different levels of difficulty so students can bite off only as much as they are prepared to chew, so to speak. Second, I have tried to split up the tasks in problems into the categories of evaluating derivatives (more straight-forward applications of rules) and finding ways to apply them (more critical thinking). This way the student can choose to work on the parts of the problems they need more work with. The more opportunities students have to teach each other and interact with peers and teachers while engaging with the material, the better!

Major Objectives
• Recall basic derivative laws: power rule, constant, e^x, sum/difference, product rule, and quotient rule. • Decompose functions using addition or multiplication principles to determine which derivative laws to use when calculating derivatives. • Apply the derivative laws to functions to find the derivative functions. • Identify complex composite functions to determine in which order to apply the chain rule. • Apply the chain rule to find a derivative. • Apply the chain rule to variables under implicit differentiation. • Analyze a problem to see how to use derivatives to solve it. • Analyze related rates problems to find the essential components.

Date	**Lesson**	**Page**
09/29	Derivatives – Trig Review • The Unit Circle • Sum Formulas • Trig Identities	28
10/01	Derivatives – Derivative Laws • Derivative Law Poster • Derivative Problems	33
10/06	Derivatives – Chain Rule • Differentiating Instruction	35
10/08	Derivatives – Implicit Differentiation • More Differentiating Instruction	39
10/13	Derivatives – Related Rates • Deconstructing Related Rates • Choose Your Own Related Rates	41

Derivatives – Trig Review September 29, 2009

Overview: Yesterday was the first exam, so we will be reviewing the exam questions for most of the class time, and then doing some trigonometry review to prepare find some trig derivates from scratch using the limit definition of the derivative.

Objectives	Assessments
• Recall angles (in radians) and their position on the unit circle along with their sine and cosine values • Use common sine and cosine values to compute uncommon values using the sum formulas • Use trigonometric identities to simplify trigonometric expressions	• Fill in a blank unit circle and a table of common sine and cosine values • Find the value of angles that are sums or differences of common angles • Simplify examples of trigonometric expressions

Vocabulary:

Radians
Unit Circle
Sum Formulas
Trigonometric Identities

Materials/Preparation:

Colored chalk
Examples to use the sum formulas and trig identities

Introduction/Hook:

Review the problems on the exam that they want to see done. It is possible they want to see all the problems, but there must be some time left at the end for trig review.

Lesson Body	Time	Inter.	Styles
<u>The Unit Circle</u>			
• Solicit student input to try to fill out the unit circle's most common angles (those with Pi over 6, 4, 3 and 2). Write them down using colored chalk, grouping all the angles with 6 in the denominator under one color, all those with 4 under another, etc.	3	M	Visual Intra.
• Briefly discuss their relationship. What kinds of patterns do they see? How do they fit together in the unit circle?	3	M	Visual Intra.
• Solicit student input to fill in the values of sin and cosine for the common angles in the first quadrant. Make sure to draw connections to the x and y values of those angles on the unit circle.	4	M	Intra. Visual
<u>Sum Formulas</u>			
• Ask students why it is not okay to distribute the sine function over addition or subtraction. Use the unit circle to show that adding two angles gives a completely different location on the unit circle, and the y-values do not simply add.	2	M	Visual
• Introduce the sum and difference formulas for the sine function. Review the symbol ±.	3	TC	Linguistic
• Put up a few examples where students have to identify the sum or difference of common angles that make up uncommon angles to find their sine values. Do not immediately show students how to go about the problem– let them talk and debate it.	5	SC	Logical Inter.
<u>Trigonometric Identities</u>			
• Drawing attention back to the unit circle, ask students what they could know about a triangle with angle theta. If none of them remember or see the	3	M	Visual Intra.

Pythagorean relationship, introduce the concept to them without giving the trigonometric identity away to them. Once they see that relationship, rewrite it and talk about the significance for any angle.			Logical
• Show briefly how to derive the other two identities from the first.	2	TC	Logical
• Have students practice using the identities in simplifying a few expressions.	5	SC	Linguistic

Wrap up:
Mention the reason why we are reviewing trig: so that we can try to find the derivative of a trigonometric function using the definition– it will involve the sum formula!

Also remind students they can make appointments if they cannot make office hours, especially to discuss the exam once they get it back.

Reflection on Lesson September 30, 2009
(Derivatives – Trig Review September 29, 2009)

General Reflections:
Some students who had been comfortable with most of the review of functions seemed to be less comfortable with trigonometry, while others who had been struggling really responded to the review. While students do not need to be completely "trig fluent", they should know that they have a list of relationships they can explore if they get stuck on a trig problem. It is an area that deserves a little more review, so I will give them a few trig problems as a warm-up next class.

Were the objectives met?
• Recall angles (in radians) and their position on the unit circle along with their sine and cosine values
Most students were familiar with these common angles, and a few of them remembered the values of the sine and cosine functions.

• Use common sine and cosine values to compute uncommon values using the sum formulas
Fewer students remembered this rule, and some were not entirely convinced why you couldn't just distribute the sine function.

• Use trigonometric identities to simplify trigonometric expressions
We did not have enough time to get to this. We will do more during the next class.

Were the assessments effective?
• Fill in a blank unit circle and a table of common sine and cosine values
Using colors helped them pick out the relationships, and showing relationships on the table helped them remember and learn new things about trig.

• Find the value of angles that are sums or differences of common angles
Students saw how to use the formula for a sum or difference of angles, but did not readily see how you could manipulate an uncommon angle and convert it to a sum or difference of common angles.

• Simplify examples of trigonometric expressions
We did not have enough time to get to this part of the lesson.

How was vocabulary introduced and reinforced?
Students seemed very comfortable with the vocabulary, which was to be expected.

Ideas for Improvement
Spending more time working with radians might help them see how we can compose or decompose common angles and uncommon angles to use the sum formulas. Doing a few independent examples in groups would be an effective way to enforce the concept.

Radians

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp (\tan x)(\tan y)}$$

$$\sin 0 = 0 = 0 = \cos \frac{\pi}{2}$$

$$\sin \frac{\pi}{6} = \frac{\sqrt{1}}{2} = \frac{1}{2} = \cos \frac{\pi}{3}$$

$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} = \cos \frac{\pi}{4}$$

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} = \cos \frac{\pi}{6}$$

$$\sin \frac{\pi}{2} = \frac{\sqrt{4}}{2} = 1 = \cos 0$$

1) $\sin\left(\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}\right) = \sin\frac{\pi}{6}\cos\frac{\pi}{4} + \cos\frac{\pi}{6}\sin\frac{\pi}{4}$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2} + \sqrt{3}\sqrt{2}}{4} = \frac{\sqrt{2}(1+\sqrt{3})}{4}$$

2) $\sin\left(\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{3}\sqrt{2} - \sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) \qquad \cos\left(\frac{7\pi}{12}\right)$$

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \qquad \cos\left(\frac{\pi}{12}\right)$$

Simplify $(\sin^2 x)(\cos^2 x) + \cos^4 x$

Prove $\cos(2x) = 2\cos^2 x - 1$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$= \cos^2 x - \sin^2 x + 1 - 1$$

$$= \cos^2 x - \sin^2 x + (\cos^2 x + \sin^2 x) - 1$$

$$= 2\cos^2 x - 1$$

Derivatives – Derivative Laws October 1, 2009

Overview: We will continue on in our trig review to get to the concepts of trig identities, and practice the sum formula a little more. Then, it is on to practice derivative laws! The challenge of forging ahead with the many derivative laws is keeping in mind that these shortcuts still represent limits, as well as slopes of tangent lines and entire functions! It will become important to use language that reminds students of these things, even while going through the milieu of examples to memorize derivative laws.

Objectives	Assessments
• Remember basic derivative laws: power rule, constant, e^x, sum/difference, product rule, and quotient rule. • Identify composite functions to determine which derivative laws to use when calculating derivatives. • Apply the derivative laws to functions to find the derivative functions.	• Derivative Law Posters • Example problems in small groups

Vocabulary:

Constant Rule	Derivative of e^x	Product Rule
Power Rule	Sum and Difference Rule	Quotient Rule

Materials/Preparation:
Post-It Posters w/ Markers
Copies of the Derivative Problems

Introduction/Hook:
Finish the trig review by doing a couple more examples of the sum formula, and then introduce the trigonometric identities. (15 minutes)

Lesson Body	Time	Inter.	Styles
<u>Derivative Law Posters</u> • To review what was done yesterday, split up into 6 groups and assign each group the task of making a poster that has one of the rules, how to apply it to a function, and an example problem. (For groups that finish early, as the constant or power rule ones may, challenge them to provide a visual example, or use the derivative definition.) • Post the finished products on the front and side walls, making sure to check them for inconsistencies.	10	SC	Inter. Visual Logical
<u>Derivative Problems</u> • In the same groups, give students a set of problems so they can practice: 1) Identifying composite functions and deciding which rule to use 2) Applying the derivative rule to find the derivative • Review the answers to these problems so students can show what they have done as well as ask questions about them.	10 10	SC M	Logical Inter. Visual

Wrap up:
Answer any homework questions they have, and offer people a chance to set up an appointment to talk about the test. Have them recall the concept of mathematical identities, and tell them to think about how their perception of their mathematical identity is affected by their test score. How will this influence what you do in the class from now on? Remind them about the flexibility of intelligence, especially with math.

Reflection on Lesson October 2, 2009
(Derivatives – Derivative Laws October 1, 2009)

General Reflections:
Trig review at the beginning was tedious for some, and way too confusing for others. There is a large gap in their comfort with trig. Some students saw places to use the trig identities and sum and double angle formulas, while others struggled to remember that sin(x) was a function. This misconception will be especially dangerous when doing the chain rule with trigonometric functions, because many students believe that sin(x) is a product.

Were the objectives met?
• Remember basic derivative laws: power rule, constant, e^x, sum/difference, product rule, and quotient rule.
Most students were starting to memorize the rules. However, some have misconceptions about the rules, especially where constants were being added or multiplied. Some students over-generalized either the constant derivative law or the constant multiple rule to both situations. For some students they didn't really understand the sum and difference rules, or how to apply multiple rules to get the derivative of a polynomial.

• Identify composite functions to determine which derivative laws to use when calculating derivatives
Students did fairly well at this, most likely because of their experiences with composite functions during the first week of the course.

• Apply the derivative laws to functions to find the derivative functions.
Most students did this fairly easily. They seemed to have a little trouble memorizing the product rule, but most of them knew how to apply it.

Were the assessments effective?
• Derivative Law Posters
This activity engaged the students about the specific rules, and they interacted a lot. They became more familiar with the rules and also went through the process of showing an example. Most of the students took it very seriously and thought hard about how to best portray their rule.

• Example problems in small groups
Most of the groups did not work together unless they were stuck on a problem. Students were able to find the derivatives individually. While it was easy to see those students who were solving the problems succeed, the ones who wanted to ask questions did not seem to feel open to asking their group members.

How was vocabulary introduced and reinforced?
The rule posters were placed on the walls for students to see. When working on problems they were able to look at them and remember the rules.

Ideas for Improvement
Using the principles of cohesive group assignments would have improved the group work that they did. While it was helpful for most of them to get some practice doing derivatives, those who were stuck did not feel safe enough to ask questions and their time was not well used. By having them make specific choices about the derivative problems (such as "Choose the best rule to use for this problem") this will encourage discussion and cohesion and allow students to feel safe asking questions. Specific choices raise the problems to a higher level of thinking and reasoning while still focusing on learning how to apply the rules.

Derivatives – Chain Rule October 6, 2009

Overview: To continue practicing the derivative laws as well as start practicing the chain rule, students will be working in groups. The principles of specific choice as well as both individual and small group work will help stimulate discussion and allow students to maximize their learning in a cooperative setting. The problems they will be working on involve doing individual work, discussing it with a partner, then the group, and making a specific decision based on the individual work.

Objectives	Assessments
• Identify composite functions to determine in which order to apply the chain rule. • Analyze a problem to see how to use derivatives to solve it. • Apply the chain rule to find a derivative.	• Individual and Partner answers • Small group answers

Vocabulary:
Chain Rule Composite Function

Materials/Preparation:
Powerpoint with group problems

Introduction/Hook:
Warm up by having students ask questions about the homework. This will give students a chance to review the mechanics of the chain rule and other derivative laws. (20 minutes)

Lesson Body	Time	Inter.	Styles
<u>Differentiating Instruction</u> • Have students in groups of 3 or 4 (4 will work best). For each problem, have students working individually or with a partner. For the second problem every two people will be working on a different part to the problem, and the group will have to piece the parts together. The group is responsible for coming up with an answer for the question.	10 10 10	SC	Intra. Inter. Logical
• For each problem, have students share their answers by soliciting methods or ideas for doing problems up on the board. For the first problem, have groups raise their hands if they chose that method. The more simultaneous reporting, the better.	5 5 5	M	Logical Visual Inter.

Wrap up:
Remind students to keep practicing using the chain rule and to come to office hours!

Reflection on Lesson October 7, 2009
(Derivatives – Chain Rule October 6, 2009)

General Reflections:
The homework questions were plentiful, and because the students were still new at all the laws, I decided to answer homework questions the whole time. This was beneficial to most students, because they did not yet have the basic familiarity with the derivative laws to try the higher-level problems. There were a few students who had already done the problems and seemed to disengage a little. For these students I tried to incorporate them as much as possible into the explanations of the solutions.

The plan is to incorporate the problems into next class as well as introduce a problem about implicit differentiation. Since we deviated from the lesson plan today (which turned out well) there is no more intentional reflection.

1) Individually, compute y' for $y = \frac{4}{(x^3+1)^2}$.

2) Share your method with a partner.
Explain it until they understand.

3) As a group, come up with at least 3 different methods total for computing the derivative.

4) Choose the best method for finding the derivative and explain the factors that influenced your decision.

Part One

How are the shapes of cubic polynomial functions related to their roots?

In other words, a cubic function can have up to 3 real roots. How would the shape and position of the graph change the number of roots the function has?

Part Two

Find the points where the function $f(x) = x^3 - 3x^2 + 3x - 6$ has horizontal tangents.

How many points are there? Based on your knowledge of cubic functions, how many points could there have been?

Part Three

Put the parts together to make a convincing explanation for this problem:

Show that $f(x) = x^3 - 3x^2 + 3x - 6$ has exactly one real root.

Part One

Find $\frac{dy}{dx}$ for the curve $x^2 + y^2 - xy + 3x - 9 = 0$.

Part Two

What is the slope of a vertical tangent line?
For example, if the derivative of a function is $\frac{x}{x-4}$,
at what point is the tangent line vertical?

Part Three

Put the parts together to solve this problem:

Find two points on the curve $x^2 + y^2 - xy + 3x - 9 = 0$ where the tangent line is vertical.

How could you differentiate the function $g(x) = ((x^2+1)^2 + (x^2+1) + 1)^2$?

Find the best way to define functions of a composite function so you could apply the Chain Rule to the function:
$g(x) = ((x^2+1)^2 + (x^2+1) + 1)^2$
Work in pairs to do so, then share with your group.

What if you had to differentiate $f(x) = (1 + (1 + (1 + x^2)^8)^8)^8$?

Derivatives – Implicit Differentiation October 8, 2009

Overview: Because the homework questions took so long last class, students will be doing the problems designed for Tuesday. There is an additional problem involving Implicit Differentiation. The problems are unfamiliar, which will give students good practice with applying some of these concepts to more difficult problems. Specifically we will be using the derivative to find tangent lines in new ways (vertical tangent lines).

Objectives	**Assessments**
• Identify composite functions to determine in which order to apply the chain rule. • Analyze a problem to see how to use derivatives to solve it. • Apply the chain rule to find a derivative. • Apply the chain rule to variables under implicit differentiation.	• Individual and Partner answers • Small group answers

Vocabulary:
Chain Rule Composite Function Implicit Differentiation

Materials/Preparation:
Powerpoint with group problems

Introduction/Hook:
Remind students that implicit differentiation simply uses the chain rule on an implicit function of x. Use the derivative of y^2 with respect to x.

Lesson Body	**Time**	**Inter.**	**Styles**
<u>Differentiating Instruction</u> • Have students in groups of 3 or 4 (4 will work best). For each problem, have students working individually or with a partner. For the second problem every two people will be working on a different part to the problem, and the group will have to piece the parts together. The group is responsible for coming up with an answer for the question.	10 10 10 10	SC	Intra. Inter. Logical
• For each problem, have students share their answers by soliciting methods or ideas for doing problems up on the board. For the first problem, have groups raise their hands if they chose that method. The more simultaneous reporting, the better.	5 5 5 5	M	Logical Visual Inter.

Wrap up:
Remind students to keep practicing using the chain rule and implicit differentiation and to come to office hours!

Reflection on Lesson October 9, 2009
(Derivatives – Implicit Differentiation October 8, 2009)

General Reflections:
Today's group work went a lot better than last week's. The students who needed more practice simply applying the rules we have been working with were able to work on the derivative parts of the problems, while students who had a handle on that were challenged by the higher level questions. There was a lot of collaboration, and even if students were stuck, most of them were motivated to figure out the problems.

Were the objectives met?
• Identify composite functions to determine in which order to apply the chain rule.
Most students seem to have a grasp on this, and many of them can see intuitively how to use the chain rule on functions without breaking them down.

• Analyze a problem to see how to use derivatives to solve it.
Most of the students also understand the connection between derivatives and slopes at this point. There was a challenging problem where they were asked to find vertical tangents, and they struggled with the idea of an undefined slope at first, but it helped solidify this concept.

• Apply the chain rule to find a derivative.
It seems like 80% of the time students can apply the chain rule correctly.

• Apply the chain rule to variables under implicit differentiation.
More students struggle with implicit differentiation, either forgetting to write the dy/dx term or else forgetting to use other rules such as the product rule correctly when it involves implicit differentiation.

Were the assessments effective?
• Individual and Partner answers
This was helpful for students who knew they needed to work on derivatives– they were able to get the help they needed in figuring out this part of problems. In general I think students were excited to work on their own time with a small group and then share their answers. They became more confident and vocal about what they knew.

• Small group answers
These problems were a little difficult for the students, but they learned a lot from the explanations, and they are similar to problems that might appear on the test, so the students gained a little confidence in working on these kinds of problems, even if they didn't immediately see how to do them.

Ideas for Improvement
A few times there were groups in which nobody really shared an idea for where to take a problem. I might try assigning groups and mixing the ability or vocal levels so that someone could get the conversation started. Also, the problems took a little longer than I planned for, and I ended up cutting them short in order to move on or get a solution everyone could ask questions about.

Derivatives – Related Rates October 13, 2009

Overview: Due to the exam tomorrow, a review of the concepts that will be on the exam would be appropriate. However, we have not reviewed related rates at all so that will take up most of the time, with little time for homework questions at the end. The motivation behind the activity in the lesson is that for students to create their own related rates problem they need to know all the components of such problems.

Objectives	**Assessments**
• Analyze related rates problems to find the essential components. • Use the components to create a related rates problem for others to solve.	• Construct a sample related rates problem together as a class. • In small groups, create a problem for other groups to solve.

Materials/Preparation:
Paper for groups to write down problems and solutions
Powerpoint with 6-step process

Lesson Body	**Time**	**Inter.**	**Styles**
<u>Deconstructing Related Rates</u> • Ask the students to brainstorm some possible situations for related rates problems. Choose one and ask them what the components of a related rates problem are. How many variables should there be? How much information must be given to solve the problem? How difficult should our relationship be?	10	SC	Intra. Logical Visual
• Solve the problem together using the 6-step process on the review sheet.	10	M	Logical
<u>Choose Your Own Related Rates</u> • In an even number of groups of 2-4 students, have students choose their own related rates problem to construct. Remind them of the essential components of a related rates problem.	10	SC	Inter. Logical Intra.
• Have groups write a nice copy of the problem and switch with another group to solve. Groups should write up their solutions in a neat, ordered format. They are encouraged to use the 6-step process, but they need not do so.	10	SC	Visual Logical Inter.

Wrap up:
Answer any homework questions they have, or review problems if there are no more homework questions, and remind them about the review session for the exam.

Reflection on Lesson October 14, 2009
(Derivatives – Related Rates October 13, 2009)

General Reflections:
Students had a difficult time imagining a related rate without my guidance, so I decided to just go through setting up an example with them. I think this was fairly valuable, even if they did not get a chance to do it themselves. The remainder of the time we spent addressing general questions and reviewing other concepts for the upcoming test.

Were the objectives met?
• Analyze related rates problems to find the essential components.
Students were mostly able to identify these essential components. They understood the basic parts but a few of them still had trouble realizing that they had to set up a relationship to take a derivative. The other thing they did not completely understand was that you have to find the derivative before substituting values.

• Use the components to create a related rates problem for others to solve.
After assessing their level of comfort with these problems, I decided that many of them were not ready to attempt this objective. Rather than struggle the day before the test I decided to try to boost their confidence by simply reviewing questions they had about solving such problems.

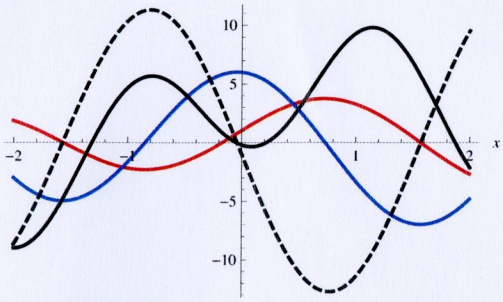
Were the assessments effective?
• Construct a sample related rates problem together as a class.
This was fairly effective. Some students were able to think of relationships and rates of change from their own Carleton examples (temperature changing, Frisbees being thrown, etc) but were unable to translate this into a related rates problem.

Ideas for Improvement
I think they were not comfortable enough with these kinds of problems to create them on their own confidently. The course simply moves too fast for some students. I am sensing increasing amounts of frustration looking at big picture concepts and theoretical problems in the course, and I think that a lot of students carry the myth that math is all about "correct" solutions to problems, instead of a set of approaches to problems that may or may not have more than one solution. As much as possible I want to instill a sense of investigation and reasoning into a course that for many students is only about algorithms and calculations. It is a difficult balance to reach to help students who carry so much anxiety over algebra that they cannot even see that they understand calculus well!

Math 101 Exam Two Fall 2009

1. (a) Suppose f is a function. Write the definition of $f'(x)$ as a limit.
   (b) Use your definition from part (a) to compute $f'(x)$ if $f(x) = \sqrt{3x+1}$.
   (c) Use our differentiation rules to compute $f'(x)$.

2. In the graph below we have $y = f(x)$, $y = f'(x)$, $y = f''(x)$, and an unrelated function $y = g(x)$. Decide which graph is which, and explain your choices.



3. For each of the following, find $\dfrac{dy}{dx}$.
   (a) $y = \sqrt{10} - \ln 5 + \sin\left(\dfrac{\pi}{7}\right) - \sqrt[3]{x}$
   (b) $y = \ln(x^3 + 3x - 1)$
   (c) $y = \dfrac{2x^2 - 1}{3e^x + 2}$
   (d) $y = 3e^x \cos(4x)$
   (e) $y = \tan^3\left(e^{x^2}\right)$

4. Find an equation for the tangent line to the curve
$$x^3 + 2x^2y + 4y^2 = 13$$
at the point $(3, -1)$.

5. Find equations for all of the tangent lines to $y = \dfrac{x+1}{x-2}$ which are parallel to the line $y + 3x = 2$.

6. The town of Park Rapids is at the intersection of highways 71 and 34, which meet at right angles. Highway 71 runs north and south, while highway 34 runs east and west. At exactly 10 pm Saturday night, Eric was 30 miles west of Park Rapids on highway 34, traveling east at 60 miles per hour. At exactly the same moment, Joel was 40 miles south of Park Rapids on highway 71, traveling south at 70 miles per hour. What was the rate of change of the distance between Eric and Joel (measured on a straight line between them) at that moment? Be sure to include appropriate units in your answer.

7. In a certain bacteria culture, the rate of growth is proportional to the number of cells present. Two hours after it begins growing, this culture contains 90 cells. Five hours after it begins growing, this culture contains 2430 cells.
   (a) How many cells will the culture contain t hours after it begins growing?
   (b) When will the culture have 10,000 cells? (You do not need to simplify your answer for this part, but you do need to include appropriate units.)

Exam One Review Problems (Chapter 3, 2.8)

Instructions: There are problems listed for practice if you aren't 100% comfortable with problems from a certain section, and then there are challenging problems that might be similar to harder ones on the test. If you don't know where to start, try the **challenging problems** first. If you are having a lot of trouble, try some of the **practice problems** from that section.

Steps for solving related rates problems:

1) Set up the problem
    a. Draw a picture to show information and relationships.
    b. Name all the variables and constants.
2) Write down numerical information
    a. Write down the values of any variables at the time in question.
    b. Write down the rates of change of any variables at the time in question.
3) Write down what you want to find
    a. Write down the answer you will want: is it a rate or a value?
    b. Write down any other values not given that you will need to find along the way.
4) Find an equation that relates the variables
    a. Based on the picture, how do all the variables relate to each other?
5) Differentiate with respect to t
    a. You will most likely need to use implicit differentiation.
    b. Remember to evaluate with respect to t (even if you have x in your problem).
6) Evaluate
    a. Substitute all the values from part 2 and see if you can solve for the information you want from part 3.

Section	Practice Problems	Challenging Problems
2.8	1, 20, 24, 41, 43	29, 48, 56
3.1	6, 8, 9, 11, 18, 36	19, 29, 46, 49, 53, 66
3.2	7, 28, 31, 43	2, 10, 11, 33, 48
3.3	8, 10, 11, 14, 24	22, 33, 42
3.4	7, 10, 21, 25, 53, 60	15, 17, 31, 42, 46, 75, 76
3.5	7, 16, 17-20, 28	6, 10
3.6	2-22, 37-48	
3.8	1, 10, 15	9
3.9	3, 5, 6, 7, 13, 14	

Exam 2 Review Answers to Even Problems

Section 2.8

20) $f'(x) = m$, domain of both is $(-\infty, \infty)$

24) $f'(x) = 1 + \frac{1}{2}x^{-\frac{1}{2}}$, domain of f is $x \geq 0$, domain of f' is $x > 0$

48) acceleration $= 0$, jerk $= -100$ ft/sec.

Section 3.1

6) $6x^7$ 8) $3t^5 - 12t^3 + 1$ 18) $\frac{1}{3}x^{-\frac{2}{3}}$ 36) $y = 12x - 3$, $y = 12x + \frac{109}{12}$

46) $G'(r) = \frac{1}{2}r^{-\frac{1}{2}} + \frac{1}{3}r^{-\frac{2}{3}}$, $G''(x) = -\frac{1}{4}r^{-\frac{3}{2}} - \frac{2}{9}r^{-\frac{5}{3}}$ 66) $y = 3x^2 - 2x + 7$

Section 3.2

2) $\frac{1}{2}x^{-\frac{1}{2}} - 3$ 10) $3u^2 + 2u + 2u^{-2}$

28) $f'(x) = e^x(\frac{5}{2}x^{\frac{3}{2}} + x^{\frac{5}{2}})$, $f''(x) = e^x(x^{\frac{5}{2}} + 5x^{\frac{3}{2}} + \frac{15}{4}x^{\frac{1}{2}})$ 48) (a) $\dfrac{3}{2}$ (b) $\dfrac{43}{12}$

Section 3.3

8) $e^u(\cos u - \sin u + cu + c)$ 10) $\dfrac{x\cos x}{(x + \cos x)^2}$ 14) $-\theta \csc\theta \cot\theta$

22) $y = x + 1$ 24) $y = -x + 1$ 42) 0

Section 3.4

10) $\frac{8}{3}x^3(1 + x^4)^{-\frac{1}{3}}$ 42) $\dfrac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-\frac{1}{2}}\left(1 + \dfrac{1}{2}(x + \sqrt{x})^{-\frac{1}{2}}(1 + \dfrac{1}{2}x^{-\frac{1}{2}})\right)$

46) $4\left(x + (x + \sin^2 x)^3\right)^3\left(1 + 3(x + \sin^2 x)^2(1 + 2\sin x\cos x)\right)$

60) $2k\pi$, $2\pi(\frac{1}{3} + k)$, $2\pi(\frac{2}{3} + k)$ for all integers k. 76) $e^{-x}(x - 1000)$

Section 3.5

6) $-2\sqrt{\dfrac{y}{x}}$ 10) $\dfrac{2xy(y^2 - e^{x^2})}{e^{x^2} - 5y^4 - 3y^2x^2}$ 16) $\dfrac{(x + y)^{-\frac{1}{2}} - 4xy^2}{4x^2y - (x + y)^{-\frac{1}{2}}}$

18) $\dfrac{\sec^2(x - y)(1 + x^2)^2 + 2xy}{(1 + x^2) + \sec^2(x - y)(1 + x^2)^2}$ 20) $\dfrac{\cos x(1 - \cos y)}{\sin y(1 - \sin x)}$ 28) $-\dfrac{x}{\sqrt{3}} + 4$

Section 3.6

2) $\dfrac{2x}{x^2 + 10}$ 4) $\dfrac{2}{\tan x}$ 6) $\dfrac{x + 1}{x\ln 5}$ 8) $\dfrac{1}{5x}$ 10) $\dfrac{2}{t(\ln t - 1)^2}$

12) $\dfrac{1}{\sqrt{x^2 - 1}}$ 14) $\dfrac{ye^y}{e^y + 1}$ 16) $\dfrac{-1}{x(\ln x)^2}$ 18) $\dfrac{z}{z^2 - a^2} - \dfrac{z}{z^2 + a^2}$

20) $\dfrac{2e^x\ln(e^x + 1)}{e^x + 1}$ 22) $\dfrac{\pi\sin\pi x - \cos\pi x}{\ln 2\cos\pi x}$ 38) $\dfrac{(x^2 + 1)^9(4x^4 + 45x^2 + 1)e^{x^2}\sqrt{x}}{2x}$

40) $\dfrac{-16x(x^2 + 1)^3}{(x^2 - 1)^5}$ 42) $\left(\dfrac{\cos x}{x} - \ln\sin x\right)x^{\cos x}$ 44) $\left(\dfrac{\ln x + 1}{2}\right)\sqrt{x}^x$

46) $\dfrac{\sin x^{\ln x - 1}(\sin x \ln \sin x + x \ln x \cos x)}{x}$

48) $\left[\dfrac{\cos x}{x \ln x} - \sin x \ln(\ln x)\right] \ln x^{\cos x}$

<u>Section 3.8</u>

10) (a) $\dfrac{\ln 0.5}{\ln 0.945}$ (b) $\dfrac{\ln 0.2}{\ln 0.945}$

<u>Section 3.9</u>

6) 25600π mm^3/s

14) $215\sqrt{101}$ km/h

Concluding Comments

Unit Two – Finding Derivatives

Effectiveness of Methods:

• Cohesive Group Assignments: Splitting up the tasks involved in solving problems, as a group seemed to be the most effective change. Students who needed more help getting down the basic laws of evaluating derivatives were able to work together to get practice doing so, and students who were ready to look at the bigger picture with derivatives were able to do so and get into some good discussions. There were still some issues when some groups made very little progress and had to be motivated to actually discuss the problems.

• Active Participation: Some of the effective techniques for engaging students during class were: asking students to draw shapes of graphs in the air using their fingers (activating their spatial *and* kinesthetic intelligences), having them discuss part of a problem with a partner, and showing them "incorrect" solutions and having them find the errors.

Ideas for Improvement:

Trigonometry Review: Because of the wide range of comfort with trigonometry, it might be better to make an optional session for those least comfortable with the concepts. That way the mandatory class can be devoted to applications of trigonometry in calculus (limits and derivatives, using identities to solve trig problems). Also, there are a lot of good charts and exercises in Appendix D of the textbook that we never used!

Unit Three – Applications of Derivatives, Integrals

Changes Made from Feedback on Midterm Evaluation: The feedback from the evaluations was mixed; half the class seemed to appreciate a slower pace, while the other half was getting bored with the review sessions. I think the class has improved since differentiating the problems for group work, but this only works well with applications of concepts we learn. Thus, when working with new concepts it is difficult to appeal to both accomplished and struggling students. As it turns out, the students really want to be able to review more homework problems. I have tried to minimize doing this for two reasons: It can be hard to learn a concept if you rely on just copying down a solution in class, and for many students reviewing homework does not engage them because they have already finished the assignment.

One solution to attempt to meet everyone's needs is to set up a system for students to choose which problems they want to ask questions on, and then as a class examine problems similar to the homework problems, but differing only slightly. This way, struggling students will learn in class and then apply those concepts later when they do the actual homework, and students who have already done the homework can practice on a fresh problem. Assuming this will take half the class time to do 3-5 problems, the rest of the time can be spent looking at more example problems that will be new to everyone and be similar in difficulty to the hardest test problems.

With this change in structure and in the interest of time, I will spend more time preparing problems to work on in class and less time preparing activities or group problems. While this new structure has active participation built into it, I will need to be creative in order to keep incorporating cooperative learning.

Major Objectives
• Use the Mean Value Theorem to find the point(s) c at which the tangent line to a function has a certain slope. • Use the First Derivative Test and Second Derivative Test to find the extrema of a function. • Use various techniques of calculus to sketch a given curve (extrema, inflection points, asymptotes). • Find an optimal solution using the First Derivative test on a particular function. • Find specific antiderivatives of a given function using given information. • Translate a given series of numbers into sigma notation or find the sum of a series written in sigma notation • Approximate the area under a curve using a variety of methods (right or left endpoint, midpoint, etc.) • Use the Riemann Sum/Limit definition of an integral to calculate the area under a curve • Use geometric manipulation to calculate the area in a given region • Use the FTC to calculate the area under a curve on a given interval

Date	Lesson	Page
10/15	Midterm Evaluation	50
10/20	Derivatives – Shapes of Graphs	55
10/22	Derivatives – Curve Sketching, Optimization	56
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11/03	Integrals – Definite Integrals, FTC	59

Derivatives – Midterm Evaluation October 15, 2009

Overview: Most of today will be spent reviewing the test problems. However, I do want to take the opportunity to stimulate some goal reviewing and setting in both my teaching and in the students' learning. Thus, we will be looking at some tips for identifying studying weaknesses and addressing ways to improve during the second half of the course. For the students this will take the form of a study skills inventory, and for myself this takes the form of a midterm evaluation. There will be no formal objectives or assessments, other than to stimulate this thinking and of course, answer some of the questions about the test problems.

Materials/Preparation:
Copies of the Study Skills Inventory
Copies of the Midterm Evaluation

Wrap up:
Remind students once again that they are welcome to set up a one-on-one meeting with any of the resources available (Eric, Erica and me) to go over a test or else think about how to set some goals for improvement.

Reflection on Lesson October 16, 2009
(Derivatives – Midterm Evaluation October 15, 2009)

General Reflections:
There were mixed reviews of the test. In general, those students who did poorly on the test because of gaps in knowledge of algebra and functions did better on this test, which tested new material, while those students who did better on the review material found the new test harder.

The responses from the evaluation were positive for the most part, and there were some interesting results. Roughly half of the students said the course moved too slowly while the other half advocated for slowing down and explaining steps more. These contradictory wishes show a divide in the students in the course that cannot be addressed under the current structure of addressing homework problems and working on the same new problems as a class. In general students appreciated group work, but they also wanted more time to ask specific questions about the homework. After discussing ideas with Eric and Deanna, I have set some midterm goals for the course *(see Unit Introduction)*.

Addressing Homework Questions… In Groups?
Personally I think reviewing homework problems is a waste of time when done in excess. In my experience thus far, the students who understand or already did the homework do not find the review valuable, and do not participate in the discussions of how to find solutions. Those students who do not understand the problems often times have separate issues that cannot be addressed all at the same time without all of them rigorously asking questions, and this requires more time and more preparation on their part. Additionally, unless students think very critically, they may not know how to approach the homework problem when not in class.

As a happy medium, I think it will benefit both kinds of students to use homework problems as a starting point for problems we do together in class. Perhaps some sort of system where students tell me their top 5 concerns with the material and we do problems that closely resemble their homework problems would be best. This way, the problems are new for everyone and we maximize the learning that can occur between peers. Hopefully this will encourage students with questions to ask them in a smaller group setting with their peers, and force them to attempt the homework problems later, reinforcing what was learned in class.

As for the group work, I think it would be easy enough with some preparation to choose problems to anticipate students concerns that they could work on in groups. Additionally, a few responses discussed placing students in groups according to ability. There are a couple ways this could be effective given the split in ability levels. The first way this could work is that given a set of problems with varying difficulty, students could self-segregate and form groups themselves based on the desire to work on a set of problems. This would hopefully allow students who need more practice with basic rules to get the practice they need in a low-pressure environment while encouraging students who are ready to attempt higher-level problems to do so. The second way to utilize ability in cooperative settings is to do more problems that require multiple levels of thinking, such as those used last in the last unit with the Chain Rule and Implicit differentiation. This method of cooperative learning would work best once all students are mostly comfortable with the material, whereas the first method is good for reinforcing new material or reviewing material before a test.

The last goal I have for improvement has to do with the way we review the test. Many students have complained that reviewing the test right after taking it is ineffective; they do not remember the work they did on specific problems. Eric supports reviewing after they receive their tests.

Math 101 Midterm Evaluation of Educational Associate

Instruction

How effective are the Tuesday/Thursday sessions in preparing you for homework and exams?

What are some things you like about the sessions? Be specific.

What are some ways to improve the sessions (either things Daniel can do better or things he could change about the sessions)?

Office Hours

Have you gone to Daniel, Erica, or Eric's Office Hours?
If yes, was the experience helpful? How was it helpful?

If not, what could be done differently?

Test Review

Did you attend the Review Sessions?
If yes, were they helpful? What was helpful about them?

What are some ways to improve these review sessions?

How helpful were the review problems? What sorts of problems would you like to see?

Math 101 Midterm Evaluation of Educational Associate (26 Responses)

Note: These are the compiled responses, with similar responses lumped under one phrasing and marked with a number in parentheses indicating how many responses total voiced that opinion.

Instruction
How effective are the Tuesday/Thursday sessions in preparing you for homework and exams?
(See answers below– they overlapped)

What are some things you like about the sessions? Be specific.
Reinforces what was learned the day before (5)
Time to ask questions about homework (7)
Practice with new material (4)
Splitting into groups (2)
Like doing problems similar to homework problems
I like the reviews (algebra, trig, etc.) (3)
Open for asking questions (2)

What are some ways to improve the sessions (either things Daniel can do better or things he could change about the sessions)?
Allow students to ask specific questions or choose problems from the homework (10)
Food or snacks
Review lecture from the day before
Too slow, want to do more (3)
Slow down, explain more steps (4)
Review problems with tricks to them (3)
Concentrate on larger concepts, not individual problems
Make the student try the problems by themselves after showing solution
Groups of similar skill-levels

Office Hours
If yes, was the experience helpful? How was it helpful?
Helpful for understanding problems (9)

If not, what could be done differently?
I forget when they are

Test Review
Did you attend the Review Sessions?
If yes, were they helpful? What was helpful about them?
Felt more confident
Going through problems in a large group with Erica reviewing with a smaller group

What are some ways to improve these review sessions?
Make the review more concise
Don't try to solve the most complicated problems
Don't assume that we know every step
Want a more general review of formulas and approaches (2)
Figure out the top 5 problems people want to see done

How helpful were the review problems? What sorts of problems would you like to see?
Answers to review problems earlier
Problems that are similar to test problems (2)
Makes studying seem more tangible
Variety of problems is good
Maybe show the solutions to some of the harder ones
Too complicated (2)
Shorter list
Too slow-paced

Derivatives – Shapes of Graphs October 20, 2009

Planning for Lesson

Overview: In order to address some of the feedback on the midterm evaluations, we will be introducing a new way of starting class and answering questions about the homework. We will also be reviewing the topic introduced last week of the critical points and absolute and local maxima/minima of graphs.

Homework Questions v2.0: When students enter they should write down the homework problems they are struggling with on the board. If the problem is already on the board, put a checkmark next to it. Take the top 3-5 problems and address them by doing problems that are similar to those problems (just as challenging or more challenging). This can be done as a class or in small groups depending on the problem.

Associated Objectives:
• Use the Mean Value Theorem to find the point(s) c at which the tangent line to a function has a certain slope.
• Use the First Derivative Test to find the extrema of a function.
• Use the Second Derivative Test to find the extrema of a function.

Possible HW Problem Modifications:
• Identify the "challenging" part of each function (ie square roots, composite functions) and keep that element but change the numbers.
• Split up the class and have one half do the First Derivative Test and the rest do the Second D.T.

Possible Further Problems:
• Discuss the relationship between the number of "curves" a function has on a given interval and the number of values of c you can find using the Mean Value Theorem.
• More practice with different kinds of functions! Have students begin to learn how to sketch their results.

Reflection on Lesson

General Reflection: In a few of the problems I simply switched numbers without making sure that the problems still worked! It is important to check that solutions exist in the proper domain! Also, for some problems it doesn't seem worth it to change the numbers. Either the problem doesn't change much by doing this, or else if the problem is very long it will take a lot of preparation. Chances are on long problems that even students who have done them will still want to see them done to double check their own answer.

Ideas for Improvement: I would have liked to do even more problems with the First Derivative and Second Derivative tests, exploiting the differences between the two methods. It is hard for many students to memorize each one, and even harder to see the appeal of one over another. I also think it helps the students to use graphs as much as possible to reinforce the relationships between the values of the derivatives and the shapes of graphs.

Derivatives – Curve Sketching, Optimization October 22, 2009

Planning for Lesson

Overview: We will be continuing with the structure of reviewing 3-5 homework problems and then moving on to new problems. Specifically, we are into curve sketching, where we are taking all the characteristics we know how to find about the shape of functions and putting it all together to accurately sketch various functions or curves. One big challenge is that students are still learning how to find horizontal asymptotes as well as extrema, two big parts of curve sketching. After curve sketching, optimization is a natural application of finding extrema, but because students do not have much experience setting up these kinds of problems, it seems very unfamiliar to them. It will be important to make connections between extrema and optimization problems as much as possible!

Associated Objectives:
• Use various techniques of calculus to sketch a given curve (extrema, inflection points, asymptotes).
• Identify the essential information needed to sketch a given curve (the minimum information needed to accurately sketch the graph).
• Describe a relationship between variables algebraically given certain circumstances.
• Find an optimal solution using the First Derivative test on a particular function.

Possible HW Problem Modifications:
• Identify the "challenging" part of each function (i.e. inflection points, holes or asymptotes) and keep that element but change the numbers.
• Focus on the difference between asymptotes or holes in graphs.
• Changing the functions used in optimization problems (especially in the graph problems) as well as changing numbers will give students practice setting up various problems, which is the most difficult part of the problems.

Possible Further Problems:
• Discuss why in some cases you do not need to do all the tests possible in order to sketch a graph. Include a few examples of problems in which you can tell right away what the domain is, or what an asymptote of the graph is, or that there are no inflection points.
• Compare different three-dimensional shapes whose surfaces areas are optimized by a constant volume (i.e. sphere, cone, cylinder, prism).
• Find optimization problems that use angles and trigonometry!

Reflection on Lesson

General Reflection: The problem that students struggle with the most is where to start a problem, or else where to go after certain steps in the problem. It is a little ridiculous to assume students will memorize all the steps listed in the book, even though they need to be able to call on any one of them in a given situation. As much as possible I tried to ask students to think outside the box and reason with me to find out if we needed to find certain pieces of information. Students need to get used to asking the right kinds of questions about graphs, not just know how to answer them with calculus techniques.

Ideas for Improvement: It would have been interesting to explore functions that have multiple ways of figuring out how to sketch them, and having groups work against each other to come up with unique methods. This would really illustrate the idea that all of these techniques are tools in a very large toolbox, not a set of strict procedures for sketching any graph.

Integrals – Antiderivatives October 27, 2009

Planning for Lesson

Overview: Those students who really understood the algebra behind the basic derivative laws will easily understand the basic antiderivatives, but students who simply memorized the laws will have an especially hard time going in reverse. One good way to phrase antiderivative problems is by asking what functions, when differentiated, will produce the given function. However, in order to memorize new rules such as the new product rule, it is essential that students fundamentally understand the basic derivative laws. So, review these as much as possible when going through these kinds of problems!

Associated Objectives:
• Use basic antiderivatives and algebra to find generic antiderivatives for more complex functions.
• Find specific antiderivatives of a given function using given information.

Possible HW Problem Modifications:
• Making problems more difficult and changing numbers is a good way to prevent students who understand antiderivatives from disengaging, and they will not be too difficult for students still learning about antiderivatives.

Possible Further Problems:
• When students get to a comfortable point, they can start to use more advanced reasoning to tackle problems that u-substitution would normally take care of (such as natural log problems or exponential problems).

Reflection on Lesson

General Reflection: This was fairly straightforward for students. The most common errors still come from handling constants or non-natural exponents of polynomials.

Ideas for Improvement: You literally can't do too many of these problems when students are still learning! Perhaps having a long list of them from students to choose from would be useful, because there were students who had a lot more trouble than others, and they might benefit from problems that are easier, while other students are ready to start tackling more advanced antiderivatives.

Integrals – Sigma Notation, Riemann Sums October 29, 2009

Planning for Lesson

Overview: The definition of an integral as the limit of a Riemann sum is very troublesome for students for several reasons: some do not fully understand sigma notation (in fact a handful have never seen it before), and the notation of Δx and x_i can make a function unrecognizable for some. Even if students can understand the mechanics of the notation, they may still fail to see why this notation is helpful for calculating an integral, especially if they understand antiderivatives well. So, it is essential to make sure students understand the basic components of this definition of an integral before learning the definition and using it in computations.

Associated Objectives:
• Translate a given series of numbers into sigma notation or find the sum of a series written in sigma notation
• Approximate the area under a curve using a variety of methods (right or left endpoint, midpoint, etc.)
• Use the Riemann Sum/Limit definition of an integral to calculate the area under a curve
• Given a Riemann Sum/Limit definition of an integral, find the curve and interval that it represents

Possible HW Problem Modifications:
• Change the number or size of the rectangles in a Riemann sum problem, or use a different method to pick the values.
• Change the bounds of an interval or the power of the polynomial to get students used to locating Δx and x_i in a given problem.

Possible Further Problems:
• Use different numbers of rectangles in a Riemann sum to reinforce the concept that with more rectangles you get a more accurate estimate.
• Draw connections between the limit definition of an integral and the antiderivative form, and have students discuss which they prefer in different contexts.

Reflection on Lesson

General Reflection: In order to help students understand the relationships between the parts of the Riemann Sum/Limit definition, I explained the indexing part of sigma notation in terms of a library metaphor: if you are looking for a certain book, the barcode tells you information about how to find it in the library. In the same way, if you are looking for the ith x-value, the formula $(a+i\Delta x)$ is like a barcode, telling you how to find the value you are searching for. This seemed to help many students understand why we introduce the variable i and why the expression for $f(x_i)$ is almost unrecognizable when you put x_i in the original function.

Ideas for Improvement: If we are expecting students to use sigma notation to compute an integral on a test, I think it deserves more time to be explored rather than just practiced. There are so many parts, and students have trouble on some or all of them, so putting the parts together can be nearly impossible for some students. However, it seems a little silly to spend so much time on something that we have a great shortcut for– antiderivatives. Just the sight of an integral written using sigma notation can send some students into shock– so perhaps an intermediate concept that would be easier to understand (something between a finite sum and the limit of an infinite one) could be used and then developed into the full concept.

Integrals – Definite Integrals, Fundamental Thm. of Calc. November 3, 2009

Planning for Lesson

Overview: Because this is the last day before the exam, we will be reviewing a lot of material, but focusing on the latest material involving calculating definite integrals using area formulas or the fundamental theorem of calculus. These are fairly straightforward, however students could have a hard time making the FTC make sense to them– it seems to come out of nowhere for many students.

Associated Objectives:
• Use geometric manipulation to calculate the area in a given region
• Use the FTC to calculate the area under a curve on a given interval
• Apply properties of definite integrals (change of bound formulas) to solve problems involving definite integrals that may be difficult or impossible to find directly using antiderivatives

Possible HW Problem Modifications:
• Substitute unknown constants (a, b, c, etc.) for numbers in area problems so students can generalize these problems (especially absolute value and semi-circles).

Possible Further Problems:
• Find examples of problems where the definite integral does not give the area under the curve, and discuss the differences.
• Pose several incorrect applications of properties of definite integrals and ask students to correct.

Reflection on Lesson

General Reflection: For the most part students seem to understand these concepts well. They are making connections between integrals and area, even though they are still learning how to evaluate the integrals they set up from graphs.

Ideas for Improvement: Again, students need to have a firm understanding of these ideas before moving on to area between curves and volume of solids, so doing lots of problems here rather than focusing on a few hard ones might be better. I think this would have been a good day to skip the daily homework questions and just have them work on problems from a worksheet.

√ 4) Calculate the area under the curve $f(x) = x^{\frac{1}{3}}$ from $[-2, 2]$. Why does the definite integral $\int_{-2}^{2} x^{1/3}\,dx$ not give the same value?

$$2\int_0^2 x^{1/3}\,dx = 2\left(\frac{3}{4}x^{4/3}\right]_0^2 = \frac{3}{2}(2)^{4/3} = 3(2)^{1/3}$$

1) Find the misapplication of integral laws:

$$\int_0^2 f(x) + \int_0^5 f(x) = \int_0^7 f(x)$$

$$\int_{-3}^{0} g(x) - \int_0^3 g(x) = 0$$

√ 2) $\int_1^e \frac{3x^2 + x - 2}{x^3}\,dx = \int_1^e \frac{3}{x} + \frac{1}{x^2} - \frac{2}{x^3}\,dx = \left(3\ln x - \frac{1}{x} + \frac{1}{x^2}\right]_1^e$

5) What can you say about a function if for even n, the nth derivative is e^{-x}?

MVT Find all points c in the interval $[-2, 2]$ where the tangent lines of $f(x) = \frac{4x^2}{2x-1}$ have the same slope as the line between the two endpoints. Why does the MVT fail? Which intervals will actually produce a c?

$\int_1^9 f(x)\,dx = -1$ $\int_7^9 f(x)\,dx = 5$ $\int_7^9 h(x)\,dx = 4$

a. $\int_1^9 -2f(x)\,dx$ b. $\int_7^9 [f(x) + h(x)]\,dx$

3

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$ d. $\int_9^1 f(x)\,dx$

e. $\int_1^7 f(x)\,dx$ f. $\int_9^7 [h(x) - f(x)]\,dx$

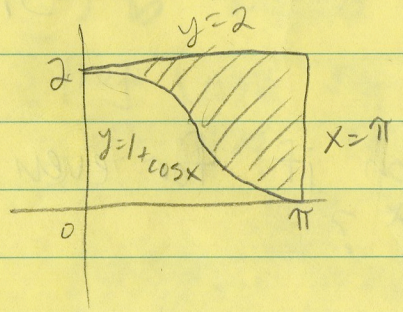
Area: $\int_{-1}^{1} (2 - |x|)\,dx$ $\int_{-1}^{1} (3 + \sqrt{2 - x^2})\,dx$

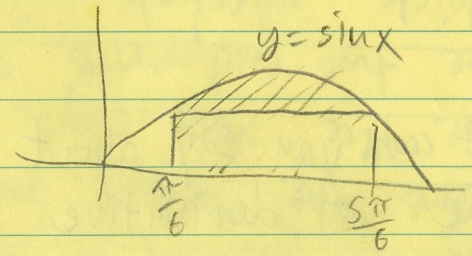
$\int_0^{\pi/3} 2\sec^2 x\,dx$ $\int_{\pi/6}^{5\pi/6} \csc^2 x\,dx$

(tan x) (−cot x)

$\int_{-4}^{4} |x|\,dx$ $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx$

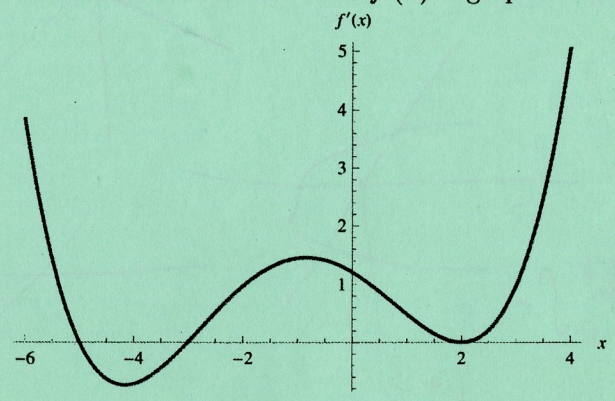
$\int_0^{\pi/2} \frac{1}{2}(2\cos x) + \int_{\pi/2}^{\pi} \frac{1}{2}(\cos x - \cos x)$





Math 101 Sample Exam Three Fall 2009

1. In this problem let f be a function whose derivative $f'(x)$ is graphed below.



   (a) On what intervals is f increasing? Explain.
   (b) On what intervals is f concave down? Explain.
   (c) Find the x-coordinates of all of the critical points of f, and classify each as a local maximum, a local minimum, or neither.

2. Find the absolute maximum and absolute minimum of $f(x) = x^2e^{-x}$ on the interval $[-1, 1]$.

3. Sketch the graph of a function $y = f(x)$ for which all of the following hold.

   (a) $f(0) = 2$.
   (b) $\lim_{x\to\infty} f(x) = -2$.
   (c) The line $y = 1$ is a horizontal asymptote for f.
   (d) $f'(x) > 0$ for all $x < 1$.
   (e) f is increasing on the interval $(1, 2)$.
   (f) $f(2) = -1$ is a local maximum for f.
   (g) f has a vertical asymptote at $x = 1$.
   (h) $f''(x) < 0$ on $(1, 3)$.
   (i) f is concave up on $(3, \infty)$.

4. In this problem let $f(x) = \dfrac{x-3}{x+2}$. Find all values of c for which the Mean Value Theorem holds for f on the interval $[0, 2]$.

5. Suppose I have a piece of wire 10 inches long, which I cut into two pieces. With one piece I make a square, and with the other I make a circle. What is the largest total area these two shapes can enclose? What is the radius of the circle in this case? Explain.

6. Find a function $g(x)$ such that $g'(x) = 3\sec^2 x - 2x^3$ and $g(0) = 4$.

7. Use the midpoint rule with $n = 4$ to approximate the area in the first octant under the curve $y = 16-x^2$. If we had used right endpoints instead of midpoints, would our approximation be an overestimate or an underestimate of the actual area? Explain.

8. Evaluate each of the following integrals.

   (a) $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

   (b) $\displaystyle\int_{-1}^2 ((2x+1)^2 + e^{-x})\,dx$

Exam Three Review Problems

4.2 - 12, 14, 15, 16
4.3 - 19, 21, 22
4.5 - Any of 1-52 (try 3-5 problems that challenge you)
4.7 - 9, 10, 13, 17, 37
4.9 - Any of 23-46 (try 3-5 problems that challenge you)
5.1 - 19, 22, 23 (a, c)
5.2 - 3, 23, 25, 40
5.3 - 19-42 (try 3-5 problems that challenge you)
App E - 21-35, 43-46

Final Exam Review Problems

I have put together a list of review problems from the end-of-chapter reviews in the book. I chose problems that are from the sections we worked on, and are fairly representative of what might be on the test. As last time, I chose some "review" problems that simply practice applications of rules or theorems, and some more challenging "test" problems that are more on the level of the test. Also, as before, there are a lot of problems. If you take time to do all of them, you will be very well-prepared. That said, if you want to focus on the areas you need the most help on, take a closer look at those chapters. One final note, if you are interested in some Trigonometry review, Appendix D has some good tables, charts and practice problems!

Chapter 1: Functions (1.1, 1.2, 1.3, 1.5, 1.6)
Review: 5-8, 25, 26 (any similar problems that you want to try!)

Chapter 2: Limits, Derivative as a limit, as a tangent (2.1, 2.2, 2.3, 2.7, 2.8)
Review: 5, 8, 10, 12, 16
Test: 2, 20, 36, 39(a,b), 45(a,b), Concept Section: 12, 13

Chapter 3: Derivative Laws, Related Rates (3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.8. 3.9)
Review: 4, 11, 17, 18, 25, 30, 18, 94
Test: 32, 60, 83, 84, 93, 98

Chapter 4: Curve Sketching, Optimization, Antiderivatives (4.1, 4.2, 4.3, 4.5, 4.7, 4.9)
Review: 3, 5, 65, 70, 72 (Note: if you don't remember the Mean Value Theorem, look at section 4.2 for some problems)
Test: 25, 28, 32, 50, 59

Chapter 5: Riemann Sums, Integrals, Substitution (5.1, 5.2, 5.3, 5.4, 5.5)
Review: 4, 17, 23, 37, 61
Test: 7, 25, 28, 58, 70

Chapter 6: Areas and Volumes with Integrals (6.1, 6.2)
Review: 2, 3, 4
Test: 8, 15, 23

Concluding Comments

Unit Three – Applications of Derivatives, Integrals

Effectiveness of Methods:

• Homework Questions v2.0: On most of the days this new system worked well as a review of concepts and an opportunity for students to ask clarifying questions. Many times, in fact, the questions led me to be able to pick better problems for them to work on during the rest of the period because I knew what their struggles were. There were couple days when students had very individualized questions or else I could not find a way to modify the problems easily, so I felt like all I did was go over homework problems that many students had already done. However, the material covered in this unit was difficult and the pace was very quick, so I think students appreciated getting a chance to see problems even if they had done them already.

• Feedback from Midterm Evaluation: I think students have adjusted well to the system– those students who had questions became more vocal about them, and the students who were struggling before seemed to become more engaged. The students who were grasping the concepts better were still engaged because they were able to get some practice on the skills they were learning.

Ideas for Improvement:

While the new system worked most of the time, I think it was too much to do every day. It wasn't that the students got tired of it– on the contrary, many of them would have gotten homework help the whole time if they could have. The issue was that some concepts (such as Riemann sums) were very difficult for students to grasp, and they weren't ready to practice doing those kinds of problems right away. If I could go back, I would have done more exploration into that whole concept before just tackling the problems. In general, the more effective use of their time might be to focus on those more difficult concepts and not worry about getting in a whole day on whatever topic they learned the day before in Eric's class.

Unit Four – Applications of Integrals

General Overview: The last unit of the course focuses on more difficult integrals as well as applications of these integrals: calculating area and volume. These problems are very difficult for many students to visualize, so my general approach is to give students as many ways to think about setting these problems up as possible. The general structure of the course is going well, so we will continue to focus on both homework problems as well as a few problems created by me for them to tackle.

One strategy I have tried with a few of the problems I have created for students to practice is to do two similar problems at a time that are different somehow. This could mean that one is easier than the other (they involve more calculations or a solution that may be difficult to see right away), or that they look at two variants on a general problem (such as two different functions being integrated in a similar way). This gives students a choice of what to work on, allowing them to get what they need from class time. Going over both solutions in front of the whole class also exposes them to more problems– even if they didn't work on one, they attempted a similar problem, so they will be more likely to understand the other solution.

Major Objectives
• Identify the expression for "u" necessary to use u-substitution to evaluate an integral • Substitute "u" and "du" and change the bounds correctly to evaluate a definite integral using u-substitution • Sketch a given region defined by two or more curves, including identifying the bounds of the region • Set up an integral that represents the area of a given region and evaluate it, including determining whether to integrate with respect to x or y • Set up and evaluate an integral to express the volume of a solid based on certain cross-sections or using the disk or washer methods

Date	Lesson	Page
11/05	Integrals – Substitution	66
11/10	Integrals – Area Between Curves	67
11/12	Integrals – Volumes	68
11/17	Integrals – Volumes, continued	68

Integrals – Substitution November 5, 2009

Planning for Lesson

Overview: Perhaps the most important thing for students to understand about the Substitution method is that it can be very hard to simply "see" which terms you will need to substitute without first making a guess and trying to find a derivative that is already in the integral. Building their confidence by doing as many problems as possible could be a good way to help them get used to exploration in this way– but doing too many problems of the same kind could lead students to focus in on a term too early for some of the problems where an initial guess for the term to substitute leads them astray.

Associated Objectives:
• Identify the expression for "u" necessary to use u-substitution to evaluate an integral
• Substitute "u" and "du" and change the bounds correctly to evaluate a definite integral using u-substitution

Possible HW Problem Modifications:
• The book has many great problems, and doing similar problems as well as some unique ones would be beneficial. In fact, going over the assigned problems could be good, especially if there are multiple ways to solve a problem.

Possible Further Problems:
• Trig problems can be especially tricky to see at first for students– and combining trig inside the argument of polynomials can be a good challenge for students.
• For now, students may think u-substitution works for any non-basic integrals. However, some integrals they will need to wait until Calc II to learn how to solve! Exposing them to this idea and giving a multiple-choice example of which integrals they could solve can help them see how substitution works in general.

Reflection on Lesson

General Reflection: Most students were able to correctly identify which terms they needed to define as "u". After making the statement that "90% of the problems will involve defining "u" as the terms inside parentheses or under a square root", we saw several problems for which this was true, but then looked at a couple that didn't work. This helped organize students' approaches to these problems a little, but they still struggled with problems with square roots and rational functions because they had a hard time identifying the pseudo-"du" terms when they were in an unexpected form (such as when the derivative of the square root of x appears as part of a denominator instead of as a negative exponent).

Ideas for Improvement: For problems that were more challenging, I wish there was a better way to help students formulate an approach to solve these. I think emphasizing the strategy of hypothesizing terms for "u" and then checking that would help more. Perhaps students could share their ideas by a vote or with a partner just to get in the habit of making an initial choice.

Integrals – Area Between Curves November 10, 2009

Planning for Lesson

Overview: The idea of approximating an area between curves is really a generalization of the area under a curve (consider the curve y=0). The big difference is when students are asked to integrate with respect to variables other than x. In order to move on to volume problems with cross-sections and rotations, students need to have an almost intuitive sense of how to set up these kinds of integrals.

Associated Objectives:
• Sketch a given region defined by two or more curves, including identifying the bounds of the region
• Set up an integral that represents the area of a given region and evaluate it, including determining whether to integrate with respect to x or y

Possible HW Problem Modifications:
• Changing numbers can be difficult because the integrals or bounds can get very difficult to integrate! Doing some of the unassigned problems could be useful, and practicing sketching ones that do not give a sketch of the region will help them learn how to set up these problems.

Possible Further Problems:
• At least one problem that could be solved easily using an integral in terms of x or y might help students see the different between the two methods. In general, it will be helpful for students to start to develop an intuition about when to use either one.

Reflection on Lesson

General Reflection: The most confusing thing to students was how to choose to integrate with respect to x or y. I tried to explain it using the terms "x-slices" and "y-slices", but I think I should have done more to explain these terms. I explained that the term "dx" referred to taking all the possible x-values, and the "x-slices" each represent an x-value. Even still, many students were confused about making this choice. Students who were confused by this also didn't make the connection that the expression inside the integral was related to these "slices". So, if you are taking "x-slices", you use "dx", your bounds should cover a range on the x-axis, and the expression must be in terms of x– many students did not seem ready to put all of these ideas together.

Ideas for Improvement: Perhaps this concept is one that could have been introduced again a second day to give students a chance to explore the ideas involved. Because it is crucial for understanding how to calculate volumes, time spent solidifying this concept would be time well spent.

Integrals – Volumes November 12 and 17, 2009

Planning for Lesson

Overview: These problems are the most difficult in terms of visualizing and setting up– they require a good understanding of what integrals represent, how they are written and expressed, and how to express values using variables in a given figure or diagram.

Associated Objectives:
• Set up and evaluate an integral to express the volume of a solid based on certain cross-sections or using the disk or washer methods.

Possible HW Problem Modifications:
• These problems are fairly difficult for students, so doing as many from the book as possible will give them an exposure to enough problems to get comfortable with them.

Possible Further Problems:
• Doing a variety of problems is good once students have the basic concepts down, but some students still struggle with visualizing these kinds of problems, much less setting them up mathematically. Thus, giving students a lot of problems to choose from might give them more freedom to practice what they want to.

Final Exam Review Answers to Even Problems

Chapter 1

6) domain: $[-2, 2]$, range: $[0, 2]$ 8) domain: $(-\infty, \infty)$, range: $[2, 4]$

26) (a) $x = \ln 5$ (b) $x = e^2$ (c) $x = \ln \ln 2$ (d) $x = \frac{\pi}{4} + k\pi$ for all integers k

Chapter 2

8) 1/3 10) -1 12) -5/54 16) 1/3

20) -1 36) when $x = 0$, $y = 6x - 4$ when $x = -1$, $y = \frac{3}{8}x + \frac{7}{8}$

Concept Section: 12) The derivative is both a limit of the slope of secant lines (which evaluates to the slope of the line tangent to $f(x)$ at $x = a$) as well as the value of a function $f'(x)$ at $x = a$.

13) The second derivative of f is the first derivative of the first derivative of f. The second derivative of the position of a particle is the first derivative of the velocity of a particle, which is equivalent to the acceleration of the particle (Recall that the third derivative is the jerk of the particle).

Chapter 3

18) $\dfrac{dy}{dx} = \dfrac{y - 2x\cos y}{2\cos 2y - x - x^2 \sin y}$ 30) $y' = \dfrac{-(x^2+1)^3(x^2+56x+9)}{(2x+1)^4(3x-1)^6}$

94) (a) 7.096g (b) 34.81 years

32) $y' = -(\sin x)e^{\cos x} - e^x \sin e^x$ 60) $y = -\frac{4}{5}x + \frac{13}{5}$, $y = \frac{5}{4}x - \frac{3}{2}$

84) (a) $y = \frac{1}{4}x + 0.597$ (b) $y = ex$ 98) 0.283 cm/sec

Chapter 4

70) $f(u) = \dfrac{u^2}{2} + 2u^{1/2} + \dfrac{1}{2}$ 72) $f(x) = \dfrac{x^5}{10} + \dfrac{x^4}{4} - \dfrac{2x^3}{3} + \dfrac{5x^2}{2} - \dfrac{251x}{60} + 2$

28) The curve is concave up from $[-\sqrt{3}, \sqrt{3}]$ and concave down otherwise. It has a minimum at $x = 0$ of 1. There are no asymptotes.

32) The curve is concave down from $[\frac{2-\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}]$ and concave up otherwise. There is a maximum at $x = 1$ of e and a horizontal asymptote at $y = 0$.

50) $x = 500$, y=125.

Chapter 5

28) $-\sin\cos x$ 58) At intervals of 1 second, estimate is 27.51m

70) Note this is the limit of a Riemann Sum, equivalent to $\int_0^1 x^9 dx$, which equals 1/10.

Chapter 6

2) 4/3 4) 32/3 8) $117\pi/5$

Teaching 101: Reflecting on Methods and Personal Growth

While the individual reflections on these lessons can offer valuable insight into how to actually run the class, many of the reflections focus on incidental experiences rather than intentional experiences. That is to say, I have learned much from teaching and reflecting on my experiences, but the most valuable things I have learned have been the result of experimenting with new ways of tackling the problems that I encountered along the way. Unfortunately, some teachers view the term "course" to mean a pathway that the class must take to get from point A to point B, so to speak. Intentional reflection, however, leads to thinking of a "course" as the journey itself, with the teacher observing, reflecting, and making several "course corrections" along the way.

First I will return to the ideas outlined in the introduction to this book relating to my own personal philosophical framework. My thoughts have changed about this framework, and I aim to show exactly how my experiences have influenced be to become a better teacher. As a result of these intentional reflections I have also gained insight into what might be missing from this framework. What follows is an effort to summarize my experiences in a coherent theory of education that bridges a gap in my framework that I never even knew existed: that gap that falsely dichotomizes "teaching" knowledge from "content" knowledge. Finally, I will comment on the structure of the class overall and my role in it so that future readers may be better prepared for such an experience.

Active Participation in Undergraduate Calculus

The following is a list of Active Participation techniques that I used in the class and played out very well. It is either the case that I didn't really have a negative experience using this framework in the course, or else I haven't noticed any negative experiences, but they did occur. To be fair, there were times when the students may have been frustrated with a certain technique for whatever reason: lack of motivation, frustration with material, discomfort speaking with a fellow student, etc. In my estimation, a positive experience with one of these techniques requires it to be effective in the sense that it successfully allows students to engage with the material. In every one of these cases, it was clear to me that those students who took advantage of these opportunities benefitted from them. They were able to articulate things they could not articulate before, they were able to see patterns that were either implicit in the dialogue or just another way of viewing a concept, and they were also able to demonstrate their understanding or lack thereof to me, so that I could better address their concerns.

- Draw this function with your finger in the air– what shape does it have? What is the limit as x approaches infinity or negative infinity?
- Consult with a partner to solve this problem (this can be any type of problem).
- Brainstorm with a partner to make a list, diagram, graph, etc.
- Fill out the unit circle (or other chart, graph, derivative, etc.) by yourself, and then be prepared to share when we go through it together (notice that this ensures that everyone has something to contribute to a discussion).
- Thumbs Up/Thumbs Down if you agree/disagree with the idea/solution proposed.

Techniques that can be improvised when students are struggling:

- Modify original homework problems so everyone can benefit from seeing them done (including students who already understand the homework).

- During the discussion of a problem as a class, if students aren't finding much to contribute, have them discuss some ideas with a partner first.

While this is not an exhaustive list of all the techniques that would be effective in any undergraduate classroom, these techniques do exhibit age-appropriate strategies for incorporating an active framework into any class. Part of the reason these specific activities worked in this course is because they supported the students in the class whose learning styles gravitated toward Kinesthetic (finding the motion of a graph with your finger), Interpersonal (Discuss with a partner), and Intrapersonal (work on a problem alone). Even though the content of the course was very much in the Logical-Mathematical category, students were able to access this knowledge through other kinds of interactions and knowledge.

Another reason these techniques were so successful was that I didn't always plan to use specific ones, but I kept them in the back of my mind, ready to use them if I realized students needed to engage more with what we were discussing in class. We did not end up using one every 3-5 minutes, but realistically we did do one every 10 minutes or so. This seemed to be effective for the undergraduate level, though (these students are more mature and self-motivated than younger students). I believe any teacher can incorporate some of these techniques into what they are already doing, simply by using them to enhance discussions when they fall flat. Active Participation is one solution to the issue of how to engage every student in the classroom, but a naturally engaging discussion can sometimes be an even better solution. However, I would rather have some tricks to pull out in a bind than have to depend on a naturally great discussion evolving.

It is interesting that the list involves so many things done with "partners". The students never had assigned partners– in fact, they tended to talk to their friends or the same people every time, which may have reduced the effectiveness of the technique. The reason this strategy worked well was because it allows students to have an informal discussion with their peers about a particular problem. Here I should emphasize that these are meant to be very short discussions; if a group of students doesn't really participate because they don't feel like they have anything to say, the activity can be counter-productive.

Collaborative Learning in Calculus

Indeed, the dynamics of group discussions or assignments can be very hard to rein in! Despite my use of the 3 S's in a few assignments during the first week, (recall from the introduction that these principles of group discussion design are: same problem, specific choice, and simultaneous reporting) I failed to realize when I innocently gave small groups some discussion questions that I had created a group-like environment with some pretty useless discussion questions. What I had not realized was that in trying to break down a more difficult problem using guiding questions, I had killed all discussion that might occur about the problems. I was not using the principle of specific choice! The questions were too simple to generate any discussion, and to be honest the experience threatened to become discouraging for me until I realized that I could have been more intentional about the questions I had produced.

The problem wasn't just one of anticipation, however. There isn't a clear or obvious way to generate productive discussions about calculus problems the same way one can generate a discussion in other disciplines. The solution I came up with seemed to be effective for generating discussion, though. This solution involved making specific choices about different parts of a problem and comparing them to other choices students could have made. For example, when working on complicated multi-step applications of derivatives, I asked students to split up into partners first, make a choice about how to solve their part of the problem, and then come back together with a small group to discuss the problem. This allowed these students to focus on the area they wanted more practice with (thus pairing up students of similar comfort level with the material) and gave them a chance to have a meaningful discussion. Later on I gave groups choices about which problems to work on in order to have them choose something at an appropriate

level, and then they had to defend their solution to the problem against the other groups who worked on that problem.

The lesson I learned about designing group discussions in calculus was valuable not only because it produced an effective method of having students collaborate, but it also made me realize just how difficult it can be to face an ineffective assignment and analyze it to come up with a solution. It wasn't until I spoke with Eric about the experience that I realized I was thinking about the assignments in a different way than I was used to thinking about group assignments. It might be too cheesy to conclude that in order to get students to collaborate, I myself had to collaborate– but it is worth noting that mathematics and teaching both have this in common!

Observing Mathematical Identities

In general, through informal assessment I have been able to see the impact I have had on students and their mathematical performance. What has been more difficult to assess has been the impact I have had on their mathematical identities. Recalling that my goal was to positively impact students and help them develop their mathematical identities, I think that for most students I was able to show them that they could be successful at math or help them see themselves as (potential) mathematicians a little more. To illustrate some of these successes I will exhibit 2 anecdotal observations (the names are fictional to protect student confidentiality).

Typical Math 101 Students

Pat first came to my office hours about a third of the way through the term, and only came a handful of times total. In class he was quiet and tended not to participate much in group discussions. He came to my office hours whenever he felt panicked about what happened in class. As we would work through a problem together, he would often say that he had no idea how to continue, and all it would take from me would be barely a hint or guiding question before he reluctantly posed a solution or next step. However, he had such a lack of confidence in his mathematical ability that he never believed he was right until I validated him. He was not alone in this thinking– many students would arrive to the correct answer just to be more confused than ever.

Terry, on the other hand, had quite the opposite mathematical identity. Terry understood most of the concepts on the first day we learned them, and seemed to thrive off of answering problems correctly in the review sessions. In fact, there were a few times in class when I would make a mistake, and Terry would be quick to point them out. I have come to understand not to take things students say personally, but it did seem strange that Terry seemed to get so frustrated if I would make a mistake. I couldn't tell at first if she was confused or if she was angry that the Educational Associate in charge had made an error! As much as possible I tried to validate her finds and make sure to comment on how good it is to check over work and that mistakes happen to everyone. As the term progressed Terry would almost laugh at some of the mistakes, and again I never could tell just why she thought to laugh.

After considering these two typical students, I realize that they have one thing in common, which is a misconception about how mathematics is done. On the one hand, students like Pat think there is typically one "right" answer, and that mathematical ability is fixed– that is, no matter how much they work they won't be able to find this answer more quickly. Students like Terry also think that there is one "right" answer, or at least that there is some mathematical currency and it is doled out by teachers to those students who can answer questions correctly. Neither student has yet come to the realization that learning mathematics involves learning a way of thinking about problems; at the core of mathematics are deductive reasoning and problem solving principles that allow mathematicians to continue to build on their knowledge and create new knowledge.

Teaching-Content Knowledge

The original framework that I described in the introduction only focused on elaborating on my teaching knowledge because I assumed no one would question the content knowledge of a senior math major in an introductory calculus course (which, after having gone through this experience, I realize can be lacking). What I didn't even realize I was assuming, however, was that teaching and content knowledge were the only necessary conditions for teaching math. What I learned after reflecting on my experiences was that I was unaware of the existence of a third kind of knowledge necessary for teaching– what I will call **teaching-content knowledge**.

While teaching knowledge can be considered general knowledge of the dynamics of learning inside the classroom, teaching-content knowledge is the content-specific knowledge of teaching methods. Different from pedagogy, this knowledge reflects the consistent struggles of students learning a certain subject in a certain approach. For example, the knowledge that students encountering the mathematical term of a "limit" may have trouble because of a pre-conceived notion of the vernacular term "limit" fits into this category. An even better example might be the difficulties students have when first learning the basic laws of derivatives– many students confuse the constant derivative law with the power rule and write the derivative of 7x as 0.

Many of these observations can be found in the reflections of these lessons, and many of those reflections come in part from my advisors on this project, Professors Eric Egge and Deanna Haunsperger. In fact, one of the most common ways teachers collaborate in the educational world is to share this type of knowledge with each other. Also, one of the primary goals for most young educators is to accumulate this knowledge through teaching courses for the first time. In fact, I believe that anyone armed with enough of this type of knowledge could teach a math course fairly well. I still believe that to make a career out of teaching (or perhaps to make a craft of it) the other types of knowledge are essential as a foundation for growing and developing. With this piece of my new framework in place to bridge the gap I never knew existed, I feel much more prepared for my future teaching endeavors, and I have a little more insight into what it is like to be an undergraduate educator.

Lessons From Lessons

The last thing I want to comment on is my analysis of the structure of the class itself. I began this project hoping to experiment with different techniques of reinforcing concepts and material, and as the term progressed I found myself with less and less time to stay ahead of the course. The format of class for the second half of the term did not help this, as my response to students' feedback was to incorporate more homework review into class. With more focus on solving specific problems, it was more difficult to continue reinforcing broad concepts without more teaching-content knowledge. Perhaps if I were to teach this course again (or someone in my place who teaches it in the future were to learn from my experiences) I would be able to anticipate more of the difficulties and take a pinpointed approach.

Again, at the core of my philosophical framework is the desire and commitment not only to improve my teaching, but also to serve my students. If I can instill this value in you, the reader, through demonstrating the insights that are attainable through such reflective practice, then I will have achieved more than I set out to do at the onset of this project. So, let these documents be a testament to my efforts and achievements, as well as an aide to anyone embarking on any similar endeavor!