Counting Pattern-Avoiding Linear Extensions of Posets

Supervisor: Eric Egge

Terms: Fall and Winter of 2015-16

Prerequisite: Previous experience with binomial coefficients, basic counting problems, and generating functions. Math 333 (Combinatorial Theory) will be sufficient, but you could also gain this experience by taking certain courses in Budapest, in an REU, or by doing some reading on your own.

Visiting Expert: Manda Riehl, University of Wisconsin at Eau Claire. Manda has done some research related to this project, and she will be visiting near the end of fall term to learn what the group has done so far, and to make suggestions about next steps and future directions.

Background Reading: *Pattern Avoidance in Extensions of Comb-Like Posets*, by Sophia Yakoubov, available on arXiv at arxiv.org/abs/1310.2979. The first thing we do will be to read this paper. You're welcome to read it ahead of time if you want, but this is not required.

Description: Suppose we have a poset P with n elements, and we number those elements from 1 to n. For example, in Figure 1 we have the Hasse diagram of the $2,3$-comb, in which $1 < 2$, $2 < 3$,

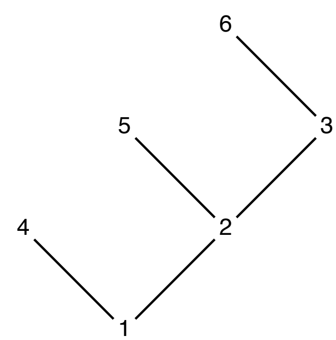

Figure 1: The $2,3$-comb

$1 < 4$, $2 < 5$, and $3 < 6$. By transitivity we also have $1 < 3$ and $1 < 6$, but 4 and 5 are not related in this poset. A *linear extension* of P is an ordering of the vertices of P (that is, a permutation of $1, 2, \ldots, n$) so that if $i < j$ in P then i appears to the left of j in the ordering. There are 15 linear extensions of the poset in Figure 1; you can see all of them in Table 1. There is a beautiful

123564	125364	123654
123546	125346	123645
123456	125436	123465
124356	124536	124365
142356	142536	142365

Table 1: Linear extensions of the $2,3$-comb

formula for the number of linear extensions of a given poset, which we may read about when we start the project, but our focus will be on counting the linear extensions which avoid a given (set of) patterns.

For any permutations π and σ, we say π *avoids* σ whenever π has no subsequence with the same length as σ, and the same relative order as σ. For example, the permutation 346251 does not avoid 213, because its subsequence 425 has the same relative ordering: largest element last, smallest element in the middle, and middle element first. On the other hand, the permutation 4132 does avoid 213, because it has no such subsequence. In the background paper listed above, Yakoubov counts linear extensions of combs which avoid various patterns of length 3. For instance, she shows that the number of linear extensions of the s,t-comb (where s and t are positive integers) which avoid 213 and 321 is the binomial coefficient $\binom{s}{2}+1$, while the number of linear extensions of the $s,2$-comb which avoid 312 is $C_{s+1}-C_s$, where C_s is the sth Catalan number.

The central goal of this project will be to extend Yakoubov's work to other posets. I have several families of candidates in mind, including rectangles of various types (illustrated in Figure 2), hooks, and triangles (illustrated in Figure 3). Although these may seem to be random choices,

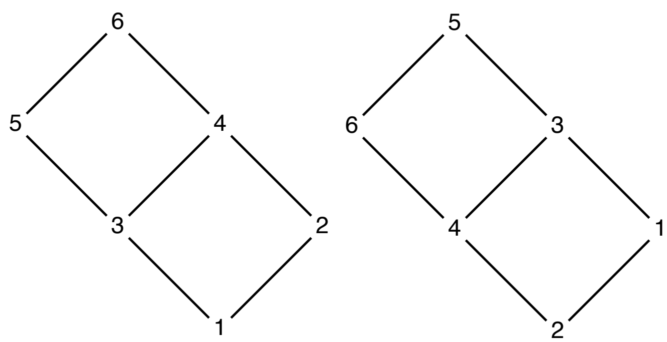

Figure 2: Two rectangles

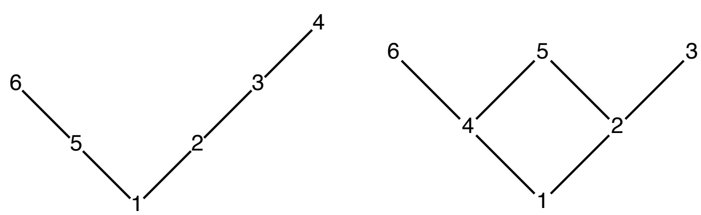

Figure 3: A hook and a triangle

they are actually motivated by their relationships with other combinatorial objects, which in turn have interesting connections with certain objects in abstract algebra. Some of our work will involve writing computer programs to generate data from which we can make conjectures (though you do not need any programming experience to participate in this project), but I expect we'll spend most of our time using combinatorial techniques like generatingfunctionology to prove (or try to prove) our conjectures.