Graph Derangements

Over the past several years Kári Ragnarsson and Bridget Tenner have studied the boolean complex of a graph \( G \), which is a certain topological space one can construct from \( G \). It turns out that the structure (more specifically, the homology) of this space can be completely described in terms of a certain set \( D(G) \) of derangements (permutations with no fixed points) one can construct explicitly from \( G \). In fact, \(|D(G)|\) already gives us useful topological information about the boolean complex. Moreover, \(|D(G)|\) is interesting in its own right for many families of graphs. For example, if \( P_n \) is the path with \( n \) vertices, then \(|D(P_n)| = F_{n-1}\), where \( F_1 = F_2 = 1 \) and \( F_n \) is the \( n \)th Fibonacci number. Equally intriguing is the fact that if \( C_n \) is the cycle with \( n \) vertices, then \(|D(C_n)| = \sum_{j=1}^{n-1} L_j - 1\), where \( L_j \) is the \( j \)th Lucas number, which may be defined by \( L_1 = 2\), \( L_2 = 1\), and \( L_n = L_{n-1} + L_{n-2} \) for \( n \geq 3 \). Also interesting is the fact that if \( K_n \) is the complete graph with \( n \) vertices, then \( D(K_n) \) is the set of all derangements of \( 1, 2, \ldots, n \).

In this project we will study \( D(G) \) for a variety of graphs \( G \). For the first two to three weeks of the project we will read some of the work of Ragnarsson and Tenner, in order to understand the construction of \( D(G) \) and survey what is already known about this set. Then we will turn our attention to some open problems involving \( D(G) \), which may include the following.

- Find \(|D(G)|\) and a combinatorial description of \( D(G) \) for other families of graphs \( G \), perhaps including triangular snakes, sparklers, caterpillars, starfish, wheels, helms, and long-armed stars.
- The set \( D(G) \) depends on a choice of labeling of the vertices of \( G \), but \(|D(G)|\) does not. In several examples the number of derangements in \( D(G) \) with a given number of cycles is also independent of the labeling. The problem is to prove this holds for some large family of graphs.
- Investigate the \( q \)-analogue of \(|D(G)|\) associated with the number of cycles statistic. For instance, if \( G \) is a path, is this one of the known \( q \)-analogues of the Fibonacci numbers?

If I can secure funding, then I will invite Bridget Tenner to visit Carleton late in the fall term, for informal conversations with this comps group about the project.

The prerequisite for this project is a course in combinatorics, either here or in Budapest.