Eric Egge

Alternating Sign Matrices, Pattern Avoidance, and Domino Tilings of Aztec Diamonds

An *alternating sign matrix* (or ASM) is a square matrix of 0s, 1s, and -1s such that

- the sum of the entries in every row and in every column is 1 and
- the nonzero entries in every row and in every column alternate in sign.

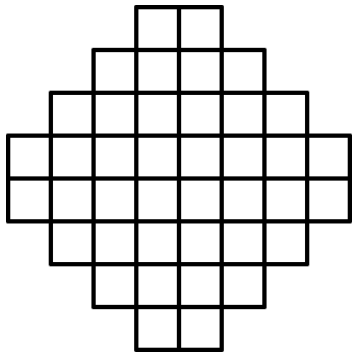
Alternating sign matrices first arose in the early 1980s, when Howard Mills, David Robbins, and David Rumsey were studying Charles Dodgson's condensation formula for determinants, but it wasn't long before they became of interest in their own right. For more than a decade one of the outstanding questions concerning ASMs was their number: Mills, Robbins, and Rumsey conjectured that there are $\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$ ASMs of dimension $n \times n$. Although Mills, Robbins, and Rumsey supported this conjecture with an even more remarkable conjecture concerting the number of $n \times n$ ASMs with a 1 in the kth position of the top row, they were unable to prove either conjecture. In 1995 Doron Zeilberger used a variety of heavy duty tools to prove Mills, Robbins, and Rumsey's formula for the number of $n \times n$ ASMs, and shortly after that Greg Kuperburg used a new connection between ASMs and the square ice model in statistical physics to give a much shorter proof of this same result. Not to be outdone, Zeilberger then used Kuperburg's methods to prove Mills, Robbins, and Rumsey's general formula for the number of $n \times n$ ASMs with a 1 in the kth position of the top row. David Bressoud tells much of this story in detail in his book *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjectures*.

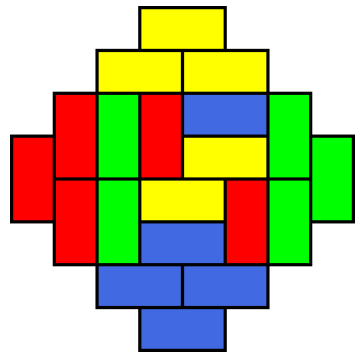
We've learned a lot about ASMs since the publication of Bressoud's book, but many unanswered questions remain; the goal of this project is to try to answer some of these more recent questions. For example, in 2007 Johansson and Linusson published a paper in which they defined pattern avoidance in ASMs, and explored some enumeration questions involving their definition. Among other things, they showed that the number of $n \times n$ ASMs which avoid the matrix corresponding with the permutation 132 is given by the $n-1$th Schröder number, a result they proved by giving a recursive bijection with Schröder paths. (The Schröder numbers are a natural analogue of the Catalan numbers; they count lattice paths from $(0,0)$ to (n,n) using only North $(0,1)$, East $(1,0)$, and diagonal $(1,1)$ steps, which do not pass below the line $y = x$.) After reading Johansson and Linusson's paper, the students in this project should be in a good position to investigate the following questions, among others.

- Can we enumerate $n \times n$ ASMs which avoid 132 and a pattern of length 4?
- Can we prove q-analogues of Johansson and Linusson's results? A natural statistic to consider on Schr̈oder paths is area, while natural statistics on ASMs include number of -1s and inversion number. In the classical permutation case there are results giving these generating functions as continued fractions, so we could look for analogues of these.
- Can we extend Johansson and Linusson's work to enumerate n x n ASMs with exactly 1 occurrence of 132? exactly 2 occurrences of 132?

Among the many mathematicians who have found connections between ASMs and other interesting objects is Jim Propp, our Chesley Lecturer. To describe a bit of this connection, you need

to know that an *Aztec diamond of order* n is an array of 1×1 boxes in which the first row has 2 boxes, the second had 4 boxes, and the kth row has $2k$ boxes for $1 \leq k \leq n$. Then the $k+1$th row also has $2k$ boxes, the $k+2$th row has $2k-2$ boxes, and the $2k$th row has 2 boxes. Further, the boxes in each row are centered. The Aztec diamond of order 4 is shown below on the left.





One of Propp's results is that the number of tilings of the Aztec diamond of order n with dominoes (one such tiling of the Aztec diamond of order 4 is shown above on the right) is $2^{\binom{n+1}{2}}$. Propp and his collaborators also showed that these tilings are bijection with ordered pairs of ASMs (one of size $n \times n$, the other of size $(n-1) \times (n-1)$) which satisfy a certain compatibility condition. In addition, Hal Canary has shown that it is possible to characterize the permutation matrices which can appear as the $n \times n$ matrix in this pair when both matrices are permutation matrices. One of the goals of this project, after understanding Canary's work, will be to study pattern-avoidance in compatible pairs of ASMs.

Since this project is closely connected with some of Jim Propp's work, I hope to capitalize on his Chesley visit in November by having him meet with the students in the project at least once, and maybe even twice. If all goes well, we will be able to share some preliminary results with Jim, and get his feedback concerning our work so far, possible next steps, and even suggestions for longer term directions to pursue.

The prerequisite for this project is a course in combinatorics, either here or in Budapest.