Solvable Leibniz Algebras with an Abelian Nilradical

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Where is Uzbekistan?
Some background on Leibniz algebras

- A Lie algebra is a vector space that has a binary bracket operation $[\cdot, \cdot]$ that abides to the following three axioms:
  - (L1) The bracket operation is bilinear.
  - (L2) $[xx] = 0$ for all $x \in L$.
  - (L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ (where $x, y, z \in L$).

- Leibniz algebras are a “noncommutative” generalization of Lie algebras.
Leibniz algebras require just bilinearity and the Leibniz identity:

$$[[a, b], c] = [a, [b, c]] + [[a, c], b].$$
Preliminaries and Definitions

Definition

A linear map \( d : L \rightarrow L \) of a Leibniz algebra \((L, [\cdot, \cdot])\) is said to be a derivation if for all \( x, y \in L \), the following condition holds:

\[
d([x, y]) = [d(x), y] + [x, d(y)].
\]

For Leibniz algebras, the right multiplication operator \( R_x : L \rightarrow L \), defined by \( R_x(y) = [y, x] \), \( y \in L \) is a derivation.

Definition

Let \( d_1, d_2, \ldots, d_n \) be derivations of a Leibniz algebra \( L \). The derivations \( d_1, d_2, \ldots, d_n \) are said to be linearly nil-independent if for \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \) and a natural number \( k \),

\[
(\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n)^k = 0 \quad \text{implies} \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.
\]
Preliminaries and Definitions

Let

\[ L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1, \]

and

\[ L^{[1]} = L, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1. \]

The following definitions will be needed:

**Definition**

A Leibniz algebra \( L \) is said to be **nilpotent** (respectively, **solvable**), if there exists \( n \in \mathbb{N} \) (\( m \in \mathbb{N} \)) such that \( L^n = 0 \) (respectively, \( L^{[m]} = 0 \)).
Preliminaries and Definitions

Definition

An ideal of a Leibniz algebra is called **nilpotent** if it is nilpotent as a subalgebra.

It is easy to see that the sum of any two nilpotent ideals is nilpotent. Therefore the maximal nilpotent ideal always exists.

Definition

*The maximal nilpotent ideal of a Leibniz algebra is said to be the nilradical of the algebra.*
A Leibniz algebra $L$ is said to be **abelian** if $L^2 = 0$. An ideal of a Leibniz algebra is called abelian if it is abelian as a subalgebra.
Problem Setup

- Let $L$ be a solvable Leibniz algebra. Then it can be written in the form $L = N \oplus Q$, where $N$ is the nilradical and $Q$ is the complementary subspace. From Casas, et al. we have:

**Theorem**

*Let $L$ be a solvable Leibniz algebra and $N$ be its nilradical. Then the dimension of $Q$ is not greater than the maximal number of nil-independent derivations of $N$."

- It was proven in Adashev, et al. that the maximal dimension of a solvable Leibniz algebra with a $k$-dimensional abelian nilradical is $2k$. Moreover, this maximal case is classified.
Here, we have classified all solvable Leibniz algebras of dimension $2k - 1$ with abelian nilradical of dimension $k$, i.e.

$$L = N \oplus Q$$

$$2k - 1 = \text{dim}(L) = \text{dim}(N) + \text{dim}(Q) = (k) + (k - 1)$$
Let $L$ be a Leibniz algebra from the class $R(A(k), k - 1)$. Then there exists a basis \{ $e_1, e_2, e_3, \ldots, e_k, x_1, x_2, \ldots, x_{k-1}$ \} of $L$ such that the table of multiplication on this basis has the following form:

\[
\begin{align*}
  e_i x_i &= e_i + \beta_{ii} e_k & 1 \leq i \leq k - 1 \\
  e_i x_j &= \beta_{ij} e_k & 1 \leq i \leq k, 1 \leq j \leq k - 1, i \neq j \\
  x_i e_i &= \alpha_i e_i + \gamma_{i,i} e_k & 1 \leq i \leq k - 1 \\
  x_i e_j &= \gamma_{i,j} e_k & 1 \leq i \leq k - 1, 1 \leq j \leq k - 1, i \neq j \\
  x_i e_k &= \sum_{j=1}^{k} \nu_{i,j} e_j & 1 \leq i \leq k - 1 \\
  x_i x_j &= \delta_{i,j} e_k & 1 \leq i, j \leq k - 1 
\end{align*}
\]

where $\alpha_i \in \{0, -1\}$. 
Let $L$ be a Leibniz algebra from the class $R(a_k, k - 1)$ and let $\alpha_i = 0$ for $1 \leq i \leq k - 1$. Then $L$ is isomorphic to one of the following algebras:
Theorem

\[ L_1(\beta_i) : \begin{cases} 
[e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_k, x_i] = \beta_i e_k, & 1 \leq i \leq k - 1, 
\end{cases} \]

\[ L_2(\beta_i) : \begin{cases} 
[e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_k, x_i] = \beta_i e_k, & 1 \leq i \leq k - 1, \\
[x_i, e_k] = -\beta_i e_k, & 1 \leq i \leq k - 1, 
\end{cases} \]

\[ L_3(\beta_i) : \begin{cases} 
[e_1, x_1] = e_1 + \beta_1 e_k, \\
[e_i, x_i] = e_i, & 2 \leq i \leq k - 1, \\
[e_1, x_i] = \beta_i e_k, & 2 \leq i \leq k - 1, \\
[e_k, x_1] = e_k, 
\end{cases} \]

\[ L_4(\nu_i) : \begin{cases} 
[e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_k, x_1] = e_k, \\
[x_1, e_k] = -e_k, \\
[x_i, e_k] = \nu_i e_1, & 2 \leq i \leq k - 1, 
\end{cases} \]

\[ L_5(\delta_{i,j}) : \begin{cases} 
[e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[x_i, x_j] = \delta_{i,j} e_k, & 1 \leq i, j \leq k - 1. 
\end{cases} \]
Theorem

Let $L$ be a solvable Leibniz algebra from the class $R(a_k, k - 1)$ and $\alpha_i = -1$ for $1 \leq i \leq k - 1$. Then $L$ is isomorphic to one of the following algebras:
Theorem

\[ L_6(\beta_j) : \begin{cases} [e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_k, x_j] = \beta_j e_k, & 1 \leq j \leq k - 1, \\
[x_i, e_i] = -e_i, & 1 \leq i \leq k - 1, \end{cases} \]

\[ L_7(\beta_j) : \begin{cases} [e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_k, x_j] = \beta_j e_k, & 1 \leq j \leq k - 1, \\
[x_i, e_i] = -e_i, & 1 \leq i \leq k - 1, \\
[x_j, e_k] = -\beta_j e_k, & 1 \leq j \leq k - 1, \end{cases} \]

\[ L_8(\gamma_i) : \begin{cases} [e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_k, x_1] = e_k, & \\
[x_i, e_i] = -e_i, & 1 \leq i \leq k - 1, \\
[x_i, e_1] = \gamma_i e_k, & 1 \leq i \leq k - 1, \end{cases} \]

\[ L_9 : \begin{cases} [e_1, x_1] = e_1 + \beta_1 e_k, & \\
[e_i, x_i] = e_i, & 2 \leq i \leq k - 1, \\
[e_1, x_i] = \beta_i e_k, & 2 \leq i \leq k - 1, \\
[e_k, x_1] = e_k, & \\
[x_1, e_1] = -e_1 - \beta_1 e_k, & \\
[x_i, e_i] = -e_i, & 2 \leq i \leq k - 1, \\
[x_i, e_1] = -\beta_i e_k, & 1 \leq i \leq k - 1, \\
[x_1, e_k] = -e_k. & \end{cases} \]

\[ L_{10}(\delta_{i,j}) : \begin{cases} [e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[x_i, e_i] = -e_i, & 1 \leq i \leq k - 1, \\
[x_i, x_j] = \delta_{i,j} e_k, & 1 \leq i, j \leq k - 1. \end{cases} \]
Theorem

Let $L$ be a solvable Leibniz algebra from the class $R(a_k, k - 1)$ and let $\alpha_1 = \cdots = \alpha_{t-1} = -1$ and $\alpha_t = \cdots = \alpha_{k-1} = 0$. Then $L$ is isomorphic to one of the following algebras:
Theorem

\[ M_{1,t}(\beta_i) : \begin{cases} 
[e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_k, x_i] = \beta_i e_k, & 1 \leq i \leq k - 1, \\
x_i, e_i] = e_i, & 1 \leq i \leq t - 1,
\end{cases} \quad M_{2,t}(\beta_i) : \begin{cases} 
[e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_k, x_i] = \beta_i e_k, & 1 \leq i \leq k - 1, \\
x_i, e_i] = -\beta_i e_k, & 1 \leq i \leq k - 1, \\
x_i, e_i] = -e_i, & 1 \leq i \leq t - 1,
\end{cases} \]

\[ M_{3,t}(\beta_i) : \begin{cases} 
[e_t, x_t] = e_t + \beta_t e_k, \\
[e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_t, x_i] = \beta_i e_k, & 1 \leq i \leq k - 1, \\
[e_k, x_t] = e_k, \\
x_i, e_i] = -e_i, & 1 \leq i \leq t - 1,
\end{cases} \quad M_{4,t}(\beta_i) : \begin{cases} 
[e_1, x_1] = e_1 + \beta_1 e_k, \\
[e_i, x_i] = e_i, & 2 \leq i \leq k - 1, \\
[e_1, x_i] = \beta_i e_k, & 2 \leq i \leq k - 1, \\
[e_k, x_1] = e_k, \\
x_1, e_1] = -e_1 - \beta_1 e_k, \\
x_i, e_i] = -e_i, & 2 \leq i \leq t - 1, \\
x_i, e_1] = -\beta_i e_k, & 2 \leq i \leq k - 1, \\
x_1, e_k] = -e_k,
\end{cases} \]

\[ M_{5,t}(\gamma_i) : \begin{cases} 
[e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_k, x_1] = e_k, \\
x_i, e_i] = e_i, & 1 \leq i \leq t - 1, \\
x_i, e_1] = \gamma_i e_k, & 2 \leq i \leq k - 1,
\end{cases} \quad M_{6,t}(\nu_i) : \begin{cases} 
[e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
[e_k, x_t] = e_k, \\
x_i, e_i] = -e_i, & 1 \leq i \leq t - 1, \\
x_t, e_k] = -e_k, \\
x_i, e_k] = \nu_i e_t, & 1 \leq i \leq k - 1,
\end{cases} \]

\[ M_{7,t}(\delta_{i,j}) : \begin{cases} 
[e_i, x_i] = e_i, & 1 \leq i \leq k - 1, \\
x_i, e_j] = -e_j, & 1 \leq i \leq t - 1, \\
x_i, x_j] = \delta_{i,j} e_k, & 1 \leq i, j \leq k - 1.
\end{cases} \]
References


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Thank you for your attention!