Combinatorial Model of Quantum Skew Symmetric Matrices

N. Eleanor Campbell

Carleton College

Quantum Points

Definition

A point (x, y) is called Quantum if x and y satisfy the commutative relation:

$$xy = qyx$$

Quantum Matrices

Definition

A 2×2 matrix is Quantum if both columns and both rows are quantum points.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$ab = qba$$
$$cd = qdc$$
$$ac = qca$$
$$bd = qdb$$

Quantum Matrices

Definition

A matrix is Quantum if all of its 2×2 sub-matrices are quantum.

Skew Symmetric Matrices

$$\begin{bmatrix} 0 & y_{21} & y_{31} & \cdots & y_{n1} \\ -y_{21} & 0 & y_{32} & \cdots & y_{n2} \\ -y_{31} & -y_{32} & 0 & \cdots & y_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -y_{n1} & -y_{n2} & -y_{n3} & \cdots & 0 \end{bmatrix}$$

The Algebra $\mathcal{O}_q(Sk_n)$

- The algebra generated by the entries in an $n \times n$ Quantum Skew Symmetric Matrix.
- $\mathcal{O}_q(Sk_n)$ is the algebra generated by some elements

$$\{y_{ij} \mid i < j \leq n\}$$

which satisfy the relations on the next slide.

Relations

$$
\begin{aligned}
y_{ij}y_{il} &= q y_{il}y_{ij} && \text{for } i<j<l \\
y_{ij}y_{jl} &= q y_{jl}y_{ij} && \text{for } i<j<l \\
y_{ij}y_{kj} &= q y_{kj}y_{ij} && \text{for } i<k<j \\
y_{ij}y_{kl} &= y_{kl}y_{ij} && \text{for } i<k<l<j \\
y_{ij}y_{kl} &= y_{kl}y_{ij} + (q-q^{-1})y_{il}y_{kj} && \text{for } i<k<j<l \\
y_{ij}y_{kl} &= y_{kl}y_{ij} + (q-q^{-1})y_{ik}y_{jl} && \\
&\qquad - q(q-q^{-1})y_{il}y_{jk} && \text{for } i<j<k<l
\end{aligned}
$$

Goal

Find a combinatorial model for $\mathcal{O}_q(Sk_n)$.

Our Model: The algebra A_n

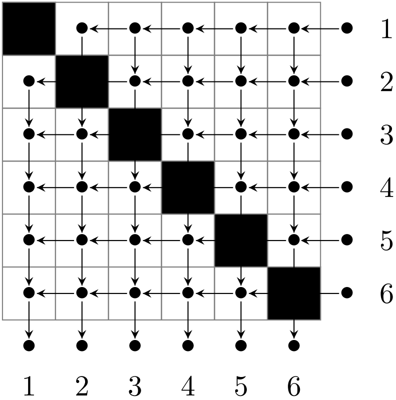


Figure: This graph gives A_6.

- Let x_{ij} be the sum of the weights of paths from right vertex labeled i to bottom vertex labeled j.
- A_n is the algebra generated by the elements of

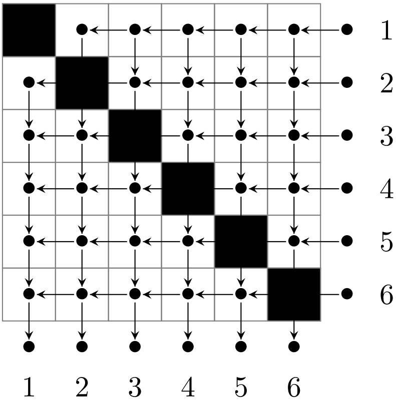
$$\{x_{ij} \mid i < j \leq n\}$$

The algebra T_n

Let T_n be the algebra generated by the elements of $\{t_{ij} \mid i < j \leq n\}$ and their inverses, which satisfy the following relations:

$$\begin{aligned} t_{ij}t_{il} &= qt_{il}t_{ij} && \text{for } i < j < l \\ t_{ij}t_{jl} &= qt_{jl}t_{ij} && \text{for } i < j < l \\ t_{ij}t_{kj} &= qt_{kj}t_{ij} && \text{for } i < k < j \\ t_{ij}t_{kl} &= t_{kl}t_{ij} && \text{for } i, j, k, l \text{ all distinct} \end{aligned}$$

Weighting Paths

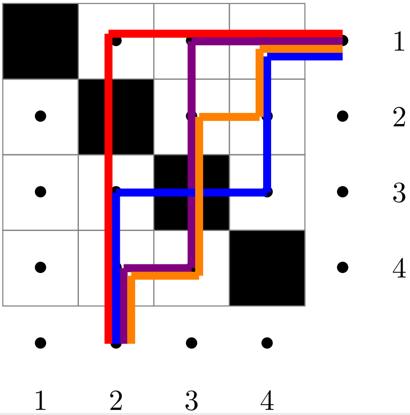


1. Begin with a weight of 1, and then consider the turns of the path in order.
2. If the path has a Γ turn at the (i, j)th vertex multiply by t_{ij}.
3. For a ┘ turn at the (i, j)th vertex multiply by t_{ij}^{-1}.
4. If the path crosses the diagonal horizontally multiply by $-q$.

Example

In A_4, here is the sum of weights of paths from 1 to 2:

$$x_{12} = t_{12} + t_{13}t_{34}^{-1}t_{24} + t_{14}t_{24}^{-1}t_{23}t_{34}^{-1}t_{24} - qt_{14}t_{34}^{-1}t_{23}$$



Theorem

$\mathcal{O}_q(Sk_n) \cong A_n$

- We define the map $\phi : \mathcal{O}_q(Sk_n) \to A_n$ to be the homomorphism which sends each y_{ij} to the corresponding x_{ij}.
- Surjectivity is given by the fact that each generator is sent to.
- To show that the map is well defined, we need to show that the x_{ij}'s satisfy the commutativity relations.
- Given well defined we can use techniques from GK-dimension theory to show injectivity.

Method

Induction on n.

Find a relation between x_{ij}'s in A_n and those in A_{n-1}.

Parent Paths

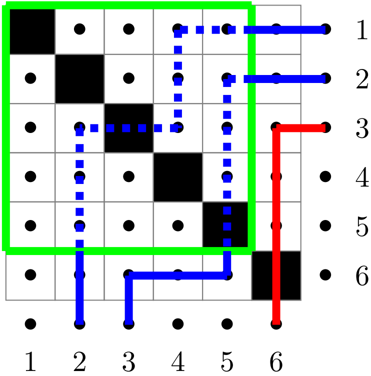
Definition

A path P_{ij} in an $n \times n$ graph has the *parent path* P_{kl} in an $n-1 \times n-1$ graph, if P_{ij} has the form:

$$P_{ij} = t_{in} t_{kn}^{-1} P_{kl} t_{ln}^{-1} t_{jn}$$

P_{ij} is then considered a *child path* of P_{kl}.

Parent Paths Graphically



- $-qt_{16}t_{16}^{-1}t_{14}t_{34}^{-1}t_{23}t_{26}^{-1}t_{26}$
- $t_{26}t_{26}^{-1}t_{25}t_{56}^{-1}t_{36}$
- t_{36}

Parental x's

$$
\begin{aligned}
x_{ij} &= \sum_{P_{i\to j}} w(P_{ij}) \\
&= \sum_{k,l} \sum_{P_{k\to l}} t_{in} t_{kn}^{-1} w(P_{kl}) t_{ln}^{-1} t_{jn} \\
&= \sum_{k,l} t_{in} t_{kn}^{-1} \left(\sum_{P_{k\to l}} w(P_{kl}) \right) t_{ln}^{-1} t_{jn} \\
&= \sum_{k,l} t_{in} t_{kn}^{-1} x_{kl} t_{ln}^{-1} t_{jn}
\end{aligned}
$$

$i \leq k < j \leq l < n$

Standard Form

$$t_{an}t_{a'n}^{-1}x_{a'b'}t_{b'n}^{-1}t_{bn}t_{cn}t_{c'n}^{-1}x_{c'd'}t_{d'n}^{-1}t_{dn}$$

Standard Form

$$t_{an}t_{a'n}^{-1}x_{a'b'}t_{b'n}^{-1}t_{bn}t_{cn}t_{c'n}^{-1}x_{c'd'}t_{d'n}^{-1}t_{dn}$$

$$t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{a'b'}x_{c'd'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$$

Standard Form

$$t_{an}t_{a'n}^{-1}x_{a'b'}t_{b'n}^{-1}t_{bn}t_{cn}t_{c'n}^{-1}x_{c'd'}t_{d'n}^{-1}t_{dn}$$

$$t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{a'b'}x_{c'd'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$$

- x's next to each other allows use of inductive hypothesis.
- Simplest rules for tracking q creation, based on the relations between indices.
- Terms from $x_{cd}x_{ab}$ can be placed in standard form without creating any q's.

Term Cancellation

$$a \leq a' < b = c \leq c' < d < b' = d'$$

- $t_{an} t^{-1}_{a'n} x_{a'b'} t^{-1}_{b'n} t_{bn} t_{cn} t^{-1}_{c'n} x_{c'd'} t^{-1}_{d'n} t_{dn}$

Term Cancellation

$$a \leq a' < b = c \leq c' < d < b' = d'$$

- $t_{an} t_{a'n}^{-1} x_{a'b'} t_{b'n}^{-1} t_{bn} t_{cn} t_{c'n}^{-1} x_{c'd'} t_{d'n}^{-1} t_{dn}$

- $t_{cn} t_{c'n}^{-1} t_{dn} t_{d'n}^{-1} x_{a'b'} x_{c'd'} t_{a'n}^{-1} t_{an} t_{b'n}^{-1} t_{bn}$

Term Cancellation

$$a \leq a' < b = c \leq c' < d < b' = d'$$

- $t_{an}t_{a'n}^{-1}x_{a'b'}t_{b'n}^{-1}t_{bn}t_{cn}t_{c'n}^{-1}x_{c'd'}t_{d'n}^{-1}t_{dn}$
- $t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{a'b'}x_{c'd'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$
- $q t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{c'd'}x_{a'b'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$

Pair Term

$$a < c \leq a' = c' < b = d \leq b'_1 < d'_1$$

Pair Term

$$a < c \leq a' = c' < b = d \leq b'_1 < d'_1$$

$$a < c \leq a' = c' < b = d \leq d'_2 < b'_2$$

$$b'_1 = d'_2 < b'_2 = d'_1$$

Pair Term Cancellation

$$a < c \leq a' = c' < b = d \leq b'_1 < d'_1$$

$$(q^3 - q) t_{cn} t_{c'n}^{-1} t_{dn} t_{d'_1 n}^{-1} x_{c' d'_1} x_{a' b'_1} t_{a'n}^{-1} t_{an} t_{b'_1 n}^{-1} t_{bn}$$

$$a < c \leq a' = c' < b = d \leq d'_2 < b'_2$$

$$(q - q^3) t_{cn} t_{c'n}^{-1} t_{dn} t_{d'_1 n}^{-1} x_{c' d'_1} x_{a' b'_1} t_{a'n}^{-1} t_{an} t_{b'_1 n}^{-1} t_{bn}$$

Cancellation Recap

Every possible ordering guarantees the existence of one of the following:

- An equal term on the opposite side of the equality.
- A pair term which, together with the original, becomes equal to two guaranteed terms on the opposite side of the equality.
- Extended versions of the second.
- Worst Case: Four terms on the left combine to equal eight on the right.

Relations

$$
\begin{aligned}
x_{ij}x_{il} &= qx_{il}x_{ij} && \text{for } i < j < l \\
x_{ij}x_{jl} &= qx_{jl}x_{ij} && \text{for } i < j < l \\
x_{ij}x_{kj} &= qx_{kj}x_{ij} && \text{for } i < k < j \\
x_{ij}x_{kl} &= x_{kl}x_{ij} && \text{for } i < k < l < j \\
x_{ij}x_{kl} &= x_{kl}x_{ij} + (q - q^{-1})x_{il}x_{kj} && \text{for } i < k < j < l \\
x_{ij}x_{kl} &= x_{kl}x_{ij} + (q - q^{-1})x_{ik}x_{jl} && \\
&\quad - q(q - q^{-1})x_{il}x_{jk} && \text{for } i < j < k < l
\end{aligned}
$$

$$\mathcal{O}_q(Sk_n) \cong A_n$$

Acknowledgments

I would like to thank my partner Phoebe Coy, our fellow group members Tamara Gomez and Chris Keane, our advisor Karel Casteels, the University of California Santa Barbara REU program, and the National Science Foundation.