Combinatorial Model of Quantum Skew Symmetric Matrices

N. Eleanor Campbell

Carleton College
Quantum Points

Definition

A point \((x, y)\) is called Quantum if \(x\) and \(y\) satisfy the commutative relation:

\[ xy = qyx \]
Quantum Matrices

Definition

A $2 \times 2$ matrix is Quantum if both columns and both rows are quantum points.

\[
\begin{bmatrix}
  a & b \\
  c & d \\
\end{bmatrix}
\]

\[
ab = qba \\
\]

\[
cd = qdc \\
\]

\[
ac = qca \\
\]

\[
bd = qdb \\
\]
Quantum Matrices

Definition

A matrix is Quantum if all of its $2 \times 2$ sub-matrices are quantum.
Skew Symmetric Matrices

$$\begin{bmatrix}
0 & y_{21} & y_{31} & \cdots & y_{n1} \\
-y_{21} & 0 & y_{32} & \cdots & y_{n2} \\
-y_{31} & -y_{32} & 0 & \cdots & y_{n3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-y_{n1} & -y_{n2} & -y_{n3} & \cdots & 0
\end{bmatrix}$$
The Algebra $O_q(Sk_n)$

- The algebra generated by the entries in an $n \times n$ Quantum Skew Symmetric Matrix.
- $O_q(Sk_n)$ is the algebra generated by some elements

$$\{y_{ij} \mid i < j \leq n\}$$

which satisfy the relations on the next slide.
Relations

\[ y_{ij}y_{il} = qy_{il}y_{ij} \quad \text{for } i < j < l \]
\[ y_{ij}y_{jl} = qy_{jl}y_{ij} \quad \text{for } i < j < l \]
\[ y_{ij}y_{kj} = qy_{kj}y_{ij} \quad \text{for } i < k < j \]
\[ y_{ij}y_{kl} = y_{kl}y_{ij} \quad \text{for } i < k < j < l \]
\[ y_{ij}y_{kl} = y_{kl}y_{ij} + (q - q^{-1})y_{il}y_{kj} \quad \text{for } i < k < j < l \]
\[ y_{ij}y_{kl} = y_{kl}y_{ij} + (q - q^{-1})y_{ik}y_{jl} - q(q - q^{-1})y_{il}y_{jk} \quad \text{for } i < j < k < l \]
Goal

Find a combinatorial model for $O_q(S_{kn})$. 
Our Model: The algebra $A_n$

Let $x_{ij}$ be the sum of the weights of paths from right vertex labeled $i$ to bottom vertex labeled $j$.

$A_n$ is the algebra generated by the elements of

$$\{x_{ij} \mid i < j \leq n\}$$

Figure: This graph gives $A_6$. 

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The algebra $T_n$

Let $T_n$ be the algebra generated by the elements of
$\{t_{ij} \mid i < j \leq n\}$ and their inverses, which satisfy the following relations:

\[
\begin{align*}
t_{ij}t_{il} &= qt_{il}t_{ij} & \text{for } i < j < l \\
t_{ij}t_{jl} &= qt_{jl}t_{ij} & \text{for } i < j < l \\
t_{ij}t_{kj} &= qt_{kj}t_{ij} & \text{for } i < k < j \\
t_{ij}t_{kl} &= t_{kl}t_{ij} & \text{for } i, j, k, l \text{ all distinct}
\end{align*}
\]
Weighting Paths

1. Begin with a weight of 1, and then consider the turns of the path in order.
2. If the path has a $\Gamma$ turn at the $(i, j)$th vertex multiply by $t_{ij}$.
3. For a $\bigcirc$ turn at the $(i, j)$th vertex multiply by $t_{ij}^{-1}$.
4. If the path crosses the diagonal horizontally multiply by $-q$. 
Example

In $A_4$, here is the sum of weights of paths from 1 to 2:

$$x_{12} = t_{12} + t_{13}t_{34}^{-1}t_{24} + t_{14}t_{24}^{-1}t_{23}t_{34}^{-1}t_{24} - qt_{14}t_{34}^{-1}t_{23}$$
We define the map $\phi: \mathcal{O}_q(Sk_n) \rightarrow A_n$ to be the homomorphism which sends each $y_{ij}$ to the corresponding $x_{ij}$.

Surjectivity is given by the fact that each generator is sent to.

To show that the map is well defined, we need to show that the $x_{ij}$'s satisfy the commutativity relations.

Given well defined we can use techniques from GK-dimension theory to show injectivity.
Method

Induction on $n$.
Find a relation between $x_{ij}$’s in $A_n$ and those in $A_{n-1}$. 

Parent Paths

**Definition**

A path $P_{ij}$ in an $n \times n$ graph has the *parent path* $P_{kl}$ in an $n-1 \times n-1$ graph, if $P_{ij}$ has the form:

$$P_{ij} = t_i t_{kn}^{-1} P_{kl} t_{ln}^{-1} t_{jn}$$

$P_{ij}$ is then considered a *child path* of $P_{kl}$. 
Parent Paths Graphically

\[ -qt_{16}t_{16}^{-1}t_{14}t_{34}^{-1}t_{23}t_{26}^{-1}t_{26} \]

\[ t_{26}t_{26}^{-1}t_{25}t_{56}^{-1}t_{36} \]

\[ t_{36} \]
Parental $x$'s

\[ x_{ij} = \sum_{P_i \rightarrow j} w(P_{ij}) \]

\[ = \sum_{k,l} \sum_{P_k \rightarrow l} t_{in} t_{kn}^{-1} w(P_{kl}) t_{ln}^{-1} t_{jn} \]

\[ = \sum_{k,l} t_{in} t_{kn}^{-1} \left( \sum_{P_k \rightarrow l} w(P_{kl}) \right) t_{ln}^{-1} t_{jn} \]

\[ = \sum_{k,l} t_{in} t_{kn}^{-1} x_{kl} t_{ln}^{-1} t_{jn} \]

\[ i \leq k < j \leq l < n \]
Standard Form

\[ t_{ant}t_{a'n}^{-1}x_{a'b'}t_{b'n}^{-1}t_{bn}t_{cn}t_{c'n}^{-1}x_{c'd'}t_{d'n}^{-1}t_{dn} \]

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Standard Form

$t_{an}^{-1} x_{a'b'} t_{bn}^{-1} t_{bn} t_{cn}^{-1} x_{c'd'} t_{dn}^{-1} t_{dn}$

$t_{cn}^{-1} t_{dn} t_{dn}^{-1} x_{a'b'} x_{c'd'} t_{an}^{-1} t_{an} t_{bn}^{-1} t_{bn}$
Standard Form

\[ t_{an} t_{a'n} x_{a'b'} t_{b'n} t_{bn} t_{cn} t_{c'n} x_{c'd'} t_{d'n} t_{dn} \]

\[ t_{cn} t_{c'n} t_{dn} t_{d'n} x_{a'b'} x_{c'd'} t_{a'n} t_{an} t_{b'n} t_{bn} \]

- x’s next to each other allows use of inductive hypothesis.
- Simplest rules for tracking q creation, based on the relations between indices.
- Terms from \( x_{cd} x_{ab} \) can be placed in standard form without creating any q’s.
Term Cancellation

\[ a \leq a' < b = c \leq c' < d < b' = d' \]

\[ t_{a'n} t_{a'n}^{-1} x_{a'b'} t_{b'n}^{-1} t_{b'n} t_{c'n} t_{c'n}^{-1} x_{c'd'} t_{d'n}^{-1} t_{d'n} t_{d'n} \]
Term Cancellation

\[ a \leq a' < b = c \leq c' < d < b' = d' \]

\[ t_{\text{ant}^{-1} t_{a'n} x_{a'b'} t_{b'n} t_{\text{cn}^{-1} t_{c'n} x_{c'd'} t_{d'n} t_{dn}} \}

\[ t_{\text{cn}^{-1} t_{c'n} t_{dn} t_{d'n} x_{a'b'} x_{c'd'} t_{a'n} t_{\text{ant}^{-1} t_{b'n} t_{bn}}} \]
Term Cancellation

\[ a \leq a' < b = c \leq c' < d < b' = d' \]

\[ t_{an}t_{a'n}^{-1}x_{a'b'}t_{b'n}^{-1}t_{bn}tc_{n}t_{c'n}^{-1}x_{c'd'}t_{d'n}^{-1}t_{dn} \]

\[ t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{a'b'}x_{c'd'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn} \]

\[ q_{tcn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{c'd'}x_{a'b'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn} \]
Pair Term

\[ a < c \leq a' = c' < b = d \leq b'_1 < d'_1 \]
Pair Term

\[ a < c \leq a' = c' < b = d \leq b'_1 < d'_1 \]

\[ a < c \leq a' = c' < b = d \leq d'_2 < b'_2 \]

\[ b'_1 = d'_2 < b'_2 = d'_1 \]
Pair Term Cancellation

\[ a < c \leq a' = c' < b = d \leq b'_1 < d'_1 \]

\[ (q^3 - q)t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{c'd'_1}x_{a'b'_1}t_{a'n}^{-1}t_{an}t_{b'_1n}^{-1}t_{bn} \]

\[ a < c \leq a' = c' < b = d \leq d'_2 < b'_2 \]

\[ (q - q^3)t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{c'd'_1}x_{a'b'_1}t_{a'n}^{-1}t_{an}t_{b'_1n}^{-1}t_{bn} \]
Cancellation Recap

Every possible ordering guarantees the existence of one of the following:

- An equal term on the opposite side of the equality.
- A pair term which, together with the original, becomes equal to two guaranteed terms on the opposite side of the equality.
- Extended versions of the second.
- Worst Case: Four terms on the left combine to equal eight on the right.
Relations

\[ x_{ij} x_{il} = qx_{il} x_{ij} \quad \text{for } i < j < l \]
\[ x_{ij} x_{jl} = qx_{jl} x_{ij} \quad \text{for } i < j < l \]
\[ x_{ij} x_{kj} = qx_{kj} x_{ij} \quad \text{for } i < k < j \]
\[ x_{ij} x_{kl} = x_{kl} x_{ij} \quad \text{for } i < k < l < j \]
\[ x_{ij} x_{kl} = x_{kl} x_{ij} + (q - q^{-1}) x_{il} x_{kj} \quad \text{for } i < k < j < l \]
\[ x_{ij} x_{kl} = x_{kl} x_{ij} + (q - q^{-1}) x_{ik} x_{jl} \]
\[ \quad - q(q - q^{-1}) x_{il} x_{jk} \quad \text{for } i < j < k < l \]
$O_q(Sk_n) \cong A_n$
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