A Borwein-Like Theorem for Overpartitions

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Definition

A partition $\lambda$ of a non-negative integer $n$ is a non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = n$. The $\lambda_i$ are called the parts of the partition. We write $p(n)$ for the number of partitions of a non-negative integer $n$. `
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Example

| $\emptyset$ | $p(0) = 1$ |
| $1$ | $p(1) = 1$ |
| $2, 11$ | $p(2) = 2$ |
| $3, 21, 111$ | $p(3) = 3$ |
Integer Partitions

Definition
A partition \( \lambda \) of a non-negative integer \( n \) is a non-increasing sequence of positive integers \( \lambda_1, \lambda_2, \ldots, \lambda_k \) such that \( \sum_{i=1}^{k} \lambda_i = n \). The \( \lambda_i \) are called the parts of the partition. We write \( p(n) \) for the number of partitions of a non-negative integer \( n \).

Example

\[
\begin{array}{ccc}
\emptyset & p(0) = 1 \\
1 & p(1) = 1 \\
2, 11 & p(2) = 2 \\
3, 21, 111 & p(3) = 3 \\
4, 31, 22, 211, 1111 & p(4) = 5 \\
\end{array}
\]
Definition

We define the $q$-Pochhammer symbol $(a; q)_\infty$ by

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$
Counting Partitions

Definition
We define the $q$-Pochhammer symbol $(a; q)\infty$ by

$$(a; q)\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

Theorem (Euler)
The generating function for the numbers $p(n)$ is given by:

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)\infty} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}$$

$$= 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots$$
Definition
An overpartition $\lambda$ of a non-negative integer $n$ is a non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = n$. The $\lambda_i$ are called the parts of the partition and the first occurrence of a part of size $j$ may be overlined. We write $\overline{p}(n)$ for the number of overpartitions of $n$. 
Definition
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Example

$\emptyset \quad \overline{p}(0) = 1$
$\overline{1}, 1 \quad \overline{p}(1) = 2$
$\overline{2}, 2, \overline{11}, 11 \quad \overline{p}(2) = 4$
$\overline{3}, 3, \overline{21}, \overline{21}, 2\overline{1}, 21, \overline{111}, 111 \quad \overline{p}(3) = 8$
Overpartitions

Definition

An overpartition $\lambda$ of a non-negative integer $n$ is a non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = n$. The $\lambda_i$ are called the parts of the partition and the first occurrence of a part of size $j$ may be overlined. We write $\overline{p}(n)$ for the number of overpartitions of $n$.

Example

$\emptyset$ $\overline{p}(0) = 1$
$\overline{1}, 1$ $\overline{p}(1) = 2$
$\overline{2}, 2, 1\overline{1}, 11$ $\overline{p}(2) = 4$
$\overline{3}, 3, 2\overline{1}, 21, 2\overline{1}, 21, 1\overline{1}1, 111$ $\overline{p}(3) = 8$
$\overline{4}, 4, 3\overline{1}, 3\overline{1}, 3\overline{1}, 31, 2\overline{2}, 22, 2\overline{1}1, 211, 2\overline{1}1, 211, 1\overline{1}11, 1111$ $\overline{p}(4) = 14$
Overpartition Generating Function

We have the generating function

\[
\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q; q)_\infty}{(q; q)_\infty}
\]

\[
= \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n}
\]

\[
= 1 + 2q + 4q^2 + 8q^3 + 14q^4 \ldots
\]
Pentagonal Numbers

Defined by \( P_n = \frac{3n^2 - n}{2} \) for any integer \( n \).

Pentagonal numbers have the useful property

\[
P_n \equiv n \pmod{3}
\]

because

\[
2P_n \equiv 3n^2 - n \pmod{3} \\
\equiv -n \pmod{3} \\
\equiv 2n \pmod{3},
\]

so \( P_n \equiv n \pmod{3} \).
Borwein’s Conjecture (the + - - - conjecture)

Given

\[ H_n(q) = \prod_{i=1}^{n} (1 - q^{3i-2})(1 - q^{3i-1}) = \sum_{j=0}^{\infty} a(j)q^j, \quad (1) \]

Conjecture

If \( a(j) \) is defined by (1), then for all \( j \geq 0 \),

\[ a(3j) \geq 0, \quad a(3j + 1) \leq 0, \quad a(3j + 2) \leq 0. \quad (2) \]
Partition Interpretation

We may consider $H_n(q)$ as a generating function for partitions into distinct parts, none of which are congruent to 0 mod 3, which subtracts the number of such partitions with an odd number of parts from those with an even number of parts.

Example

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+2  -1  -1  +3  -2  -1
Theorem

Defining the sequence $b(n)$ by

$$F(q) = \sum_{n=0}^{\infty} b(n)q^n = \prod_{k=1}^{\infty} \frac{(1 - q^{3k-1})(1 - q^{3k-2})}{(1 + q^{3k-1})(1 + q^{3k-2})}$$

we will have $b(3j + 2) = 0$ for all integers $j \geq 0$. 

Abelian Group A Borwein-Like Theorem for Overpartitions
Conjecture (+ − 0)

In the series $F(q) = \sum_{n=0}^{\infty} b(n)q^n$, we will have

$$b(3n) \geq 0, \quad b(3n + 1) \leq 0, \quad \text{and} \quad b(3n + 2) = 0$$

for all $n \geq 0$. 
Borwein-Like Theorem

Theorem (equivalent form)

Let \( n \equiv 2 \pmod{3} \). Consider overpartitions of \( n \) into parts not divisible by 3. Then the number of such overpartitions into an even number of parts equals the number of such overpartitions into an odd number of parts.
Theorem (equivalent form)

Let \( n \equiv 2 \) (mod 3). Consider overpartitions of \( n \) into parts not divisible by 3. Then the number of such overpartitions into an even number of parts equals the number of such overpartitions into an odd number of parts.

Example \((n = 5)\)

\[
\begin{align*}
5 & \quad 22\bar{1} & \quad 41 & \quad 21111 \\
\bar{5} & \quad \bar{2}2\bar{1} & \quad \bar{4}1 & \quad \bar{2}1111 \\
221 & \quad 11111 & \quad 4\bar{1} & \quad 2\bar{1}11 \\
\bar{2}21 & \quad \bar{1}1111 & \quad \bar{4}\bar{1} & \quad \bar{2}\bar{1}11 \\
\end{align*}
\]
Proof: Jacobi Triple Product

Theorem (Jacobi Triple Product)

\[ (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty} = \sum_{n \in \mathbb{Z}} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \]
Proof: Jacobi Triple Product

Theorem (Jacobi Triple Product)

\[-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty = \sum_{n \in \mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2}\]

Sketch of Proof.
We deduce via the Jacobi triple product that

\[F(q) = \frac{(q; q^3)_\infty (q^2; q^3)_\infty}{(-q; q^3)_\infty (-q^2; q^3)_\infty} = \frac{(q; q^3)_\infty (q^2; q^3)_\infty (q^3; q^3)_\infty}{(-q; q^3)_\infty (-q^2; q^3)_\infty (q^3; q^3)_\infty}\]

\[= \sum_{n \in \mathbb{Z}} (-1)^n q^{P_n} / \sum_{n \in \mathbb{Z}} q^{P_n},\]

where \(P_n = \frac{1}{2} n(3n - 1)\) is the \(n\)th pentagonal number.
Proof: Sieve

Sketch of Proof, cont.
We sieve \( F(q) \) to get \( G(q) = \sum_{n=0}^{\infty} b(3n + 2)q^{3n+2} \) as follows. Let \( \omega \) be a primitive 3rd root of unity. Then

\[
S(q) = F(q) + \omega F(\omega q) + \omega^2 F(\omega^2 q)
\]
Proof: Sieve

**Sketch of Proof, cont.**

We sieve \( F(q) \) to get \( G(q) = \sum_{n=0}^{\infty} b(3n+2)q^{3n+2} \) as follows. Let \( \omega \) be a primitive 3rd root of unity. Then

\[
S(q) = F(q) + \omega F(\omega q) + \omega^2 F(\omega^2 q)
\]
\[
= b(0) + b(1)q + b(2)q^2 + b(3)q^3 + b(4)q^4 + b(5)q^5 + \cdots
\]
\[
+ \omega b(0) + \omega^2 b(1)q + b(2)q^2 + \omega b(3)q^3 + \omega^2 b(4)q^4 + b(5)q^5 + \cdots
\]
\[
+ \omega^2 b(0) + \omega b(1)q + b(2)q^2 + \omega^2 b(3)q^3 + \omega b(4)q^4 + b(5)q^5 + \cdots
\]
Sketch of Proof, cont.

We sieve $F(q)$ to get $G(q) = \sum_{n=0}^{\infty} b(3n + 2)q^{3n+2}$ as follows. Let $\omega$ be a primitive 3rd root of unity. Then

$$S(q) = F(q) + \omega F(\omega q) + \omega^2 F(\omega^2 q)$$

$$= b(0) + b(1)q + b(2)q^2 + b(3)q^3 + b(4)q^4 + b(5)q^5 + \cdots$$

$$+ \omega b(0) + \omega^2 b(1)q + b(2)q^2 + \omega b(3)q^3 + \omega^2 b(4)q^4 + b(5)q^5 + \cdots$$

$$+ \omega^2 b(0) + \omega b(1)q + b(2)q^2 + \omega^2 b(3)q^3 + \omega b(4)q^4 + b(5)q^5 + \cdots$$

$$= 0b(0) + 0b(1)q + 3b(2)q^2 + 0b(3)q^3 + 0b(4)q^4 + 3b(5)q^5 + \cdots.$$
Proof: Sieve

Sketch of Proof, cont.

We sieve $F(q)$ to get $G(q) = \sum_{n=0}^{\infty} b(3n + 2)q^{3n+2}$ as follows.
Let $\omega$ be a primitive 3rd root of unity. Then

\[
S(q) = F(q) + \omega F(\omega q) + \omega^2 F(\omega^2 q)
\]
\[
= b(0) + b(1)q + b(2)q^2 + b(3)q^3 + b(4)q^4 + b(5)q^5 + \cdots
+ \omega b(0) + \omega^2 b(1)q + b(2)q^2 + \omega b(3)q^3 + \omega^2 b(4)q^4 + b(5)q^5 + \cdots
+ \omega^2 b(0) + \omega b(1)q + b(2)q^2 + \omega^2 b(3)q^3 + \omega b(4)q^4 + b(5)q^5 + \cdots
= 0b(0) + 0b(1)q + 3b(2)q^2 + 0b(3)q^3 + 0b(4)q^4 + 3b(5)q^5 + \cdots
= 3G(q).
\]
Proof: Denominator is Irrelevant

Sketch of Proof, cont.

Writing

\[ F(q) = \frac{\sum_{n \in \mathbb{Z}} (-1)^n q^P_n}{\sum_{n \in \mathbb{Z}} q^P_n} = \frac{A(q)}{B(q)}, \]

we get

\[ G(q) = \frac{A(q)B(\omega q)B(\omega^2 q) + \omega A(\omega q)B(\omega^2 q)B(q) + \omega^2 A(\omega^2 q)B(q)B(\omega q)}{3B(q)B(\omega q)B(\omega^2 q)}. \]
Proof: Denominator is Irrelevant

Sketch of Proof, cont.

Writing

\[ F(q) = \frac{\sum_{n \in \mathbb{Z}} (-1)^n q^{P_n}}{\sum_{n \in \mathbb{Z}} q^{P_n}} = \frac{A(q)}{B(q)}, \]

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\[ G(q) = \frac{A(q)B(\omega q)B(\omega^2 q) + \omega A(\omega q)B(\omega^2 q)B(q) + \omega^2 A(\omega^2 q)B(q)B(\omega q)}{3B(q)B(\omega q)B(\omega^2 q)}. \]

We see that the numerator is a \( q^{3n+2} \) sieve of \( 3A(q)B(\omega q)B(\omega^2 q) \), so the coefficients of \( q^{3n} \) and \( q^{3n+1} \) will be 0. If we can show that the numerator is in fact zero, then \( G(q) \) will be zero as well.
Proof: Closed Form of Coefficients

Sketch of Proof, cont.

We find that the numerator is equal to \( \sum_{N=0}^{\infty} R(N)q^N \), where

\[
R(N) = \sum_{k,m,n \in \mathbb{Z}} \omega^{m-n} \left( (-1)^k + (-1)^m \omega + (-1)^n \omega^2 \right).
\]

We will define \( f(k, m, n) = \omega^{m-n} \left( (-1)^k + (-1)^m \omega + (-1)^n \omega^2 \right) \)
so that we may write

\[
R(N) = \sum_{k,m,n \in \mathbb{Z}} f(k, m, n).
\]
Proof: Defining a Function \( \phi \)

**Sketch of Proof, cont.**

Letting \( X = \{(k, m, n) \in \mathbb{Z}^3 \mid k + m + n \equiv 2 \pmod{3}\} \), we define a function \( \phi : X \rightarrow X \) by

\[
\phi(k, m, n) = \left( k - \frac{2s - 1}{3}, m - \frac{2s - 1}{3}, n - \frac{2s - 1}{3} \right) = (k', m', n'),
\]

where \( s = k + m + n \). This is a well-defined function, since \( 2s - 1 \equiv 2 \cdot 2 - 1 \equiv 0 \pmod{3} \) and

\[
s' = k' + m' + n' = s - (2s - 1) = 1 - s \equiv 2 \pmod{3}.
\]

We will also have \( P_k + P_m + P_n = P_{k'} + P_{m'} + P_{n'} \), although we will not show this here.
Proof: $\phi$ is an Involution

Sketch of Proof, cont.

We have

$$\phi(k, m, n) = \left( k - \frac{2s - 1}{3}, m - \frac{2s - 1}{3}, n - \frac{2s - 1}{3} \right) = (k', m', n')$$

where $s = k + m + n$. We also see that $\phi$ is an involution, since

$$\phi(k', m', n') = \left( k' - \frac{2s - 1}{3}, \ldots \right)$$

$$= \left( k - \frac{2s - 1}{3} - \frac{2(1 - s) - 1}{3}, \ldots \right)$$

$$= (k, m, n),$$

with the second and third terms analogous to the first.
Proof: $\phi$ Changes Sign of $f$

Sketch of Proof, cont.

We see that

$$\frac{2s - 1}{3} \equiv 2s - 1 \equiv 1 \pmod{2},$$

and

$$m' - n' = \left( m - \frac{2s - 1}{3} \right) - \left( n - \frac{2s - 1}{3} \right) = m - n,$$

so

$$f(\phi(k, m, n)) = \omega^{m'-n'} \left[ (-1)^{k'} + (-1)^{m'}\omega + (-1)^{n'}\omega^2 \right]$$

$$= -\omega^{m-n} \left[ (-1)^{k} + (-1)^{m}\omega + (-1)^{n}\omega^2 \right]$$

$$= -f(k, m, n).$$
Sketch of Proof, cont.

Finally, we see that

\[ R(N) = \sum f(k, m, n) \]
\[ = \frac{1}{2} \left[ \sum f(k, m, n) + \sum f(\phi(k, m, n)) \right] \]
\[ = \frac{1}{2} \left[ \sum f(k, m, n) + \sum [-f(k, m, n)] \right] \]
\[ = 0, \]

as desired. \qed
Theorem (Overpartition Nonresidue)

Given a prime \( p \), define the sequence \( b_p(n) \) by

\[
F_p(q) = \sum_{n=0}^{\infty} b_p(n)q^n
= \prod_{k=1}^{\infty} \frac{(1 - q^k)(1 + q^{kp})}{(1 + q^k)(1 - q^{kp})}
= (q; q)_{\infty}(-q^p; q^p)_{\infty}
= (q; q)_{\infty}(q^p; q^p)_{\infty}.
\]

Then when \( n \) is a quadratic nonresidue modulo \( p \), we have

\[
b_p(n) = 0.
\]
Theorem (Overpartition Nonresidue, equivalent form)

Let $n$ be a quadratic nonresidue modulo a prime $p$. Consider the overpartitions of $n$ into parts not divisible by $p$. Then the number of such overpartitions into an even number of parts equals the number of such overpartitions into an odd number of parts.
Proof (Andrews): An Identity

Sketch of Proof.
We have the identity

\[
\frac{(q; q)_\infty}{(-q; q)_\infty} = \frac{(q; q)_\infty (q; q)_\infty}{(-q; q)_\infty (q; q)_\infty} = (q; q^2)_\infty (q; q)_\infty = (q; q^2)_\infty (q^2; q^2)_\infty = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}
\]

by the Jacobi triple product.
Proof (Andrews): Expression for $b_p(n)$

Sketch of Proof, cont.

We may rewrite $F_p(q)$ as

$$\sum_{n=0}^{\infty} b_p(n)q^n = \frac{(q; q)_{\infty}(-q^p; q^p)_{\infty}}{(-q; q)_{\infty}(q^p; q^p)_{\infty}}$$

$$= \left(\sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}\right) \left(\sum_{n=0}^{\infty} g(n)q^{pn}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{-\lfloor \sqrt{n} \rfloor \leq k \leq \lfloor \sqrt{n} \rfloor} (-1)^k g\left(\frac{n-k^2}{p}\right)\right)q^n.$$
Proof (Andrews): Conclusion

Sketch of Proof, cont.

Thus

\[ b_p(n) = \sum_{-\lfloor \sqrt{n} \rfloor \leq k \leq \lfloor \sqrt{n} \rfloor} (-1)^k g \left( \frac{n - k^2}{p} \right). \]

When \( n \) is a quadratic nonresidue modulo \( p \), there will be no \( k \) so that \( p \mid n - k^2 \), so the sum will be empty and \( b_p(n) = 0. \) \qed
References

- G. E. Andrews *private correspondence*.