# A Borwein-Like Theorem for Overpartitions

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# Integer Partitions

## Definition

A **partition** $\lambda$ of a non-negative integer $n$ is a non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = n$. The $\lambda_i$ are called the **parts** of the partition. We write $p(n)$ for the number of partitions of a non-negative integer $n$.

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## Example

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|---|---|
| $\emptyset$ | $p(0) = 1$ |
| $1$ | $p(1) = 1$ |
| $2, 11$ | $p(2) = 2$ |
| $3, 21, 111$ | $p(3) = 3$ |

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| $4, 31, 22, 211, 1111$ | $p(4) = 5$ |

# Counting Partitions

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We define the $q$-Pochhammer symbol $(a;q)_\infty$ by

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### Theorem (Euler)

*The generating function for the numbers $p(n)$ is given by:*

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= \frac{1}{(q;q)_\infty} = \prod_{n=1}^{\infty}\frac{1}{1-q^n} \\ &= 1+q+2q^2+3q^3+5q^4+7q^5+\cdots\end{aligned}$$

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### Example

$$\begin{array}{ll} \emptyset & \overline{p}(0) = 1 \\ \overline{1}, 1 & \overline{p}(1) = 2 \\ \overline{2}, 2, \overline{1}1, 11 & \overline{p}(2) = 4 \\ \overline{3}, 3, \overline{2}\overline{1}, \overline{2}1, 2\overline{1}, 21, \overline{1}11, 111 & \overline{p}(3) = 8 \end{array}$$

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| $\overline{1}, 1$ | $\overline{p}(1) = 2$ |
| $\overline{2}, 2, \overline{1}1, 11$ | $\overline{p}(2) = 4$ |
| $\overline{3}, 3, \overline{2}\overline{1}, \overline{2}1, 2\overline{1}, 21, \overline{1}11, 111$ | $\overline{p}(3) = 8$ |
| $\overline{4}, 4, \overline{3}\overline{1}, \overline{3}1, 3\overline{1}, 31, \overline{2}2, 22, \overline{2}\overline{1}1, \overline{2}11, 2\overline{1}1, 211, \overline{1}111, 1111$ | $\overline{p}(4) = 14$ |

# Overpartition Generating Function

We have the generating function

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 \cdots$$

# Pentagonal Numbers

Defined by $P_n = \frac{3n^2 - n}{2}$ for *any* integer $n$.

Pentagonal numbers have the useful property

$$P_n \equiv n \pmod 3$$

because

$$\begin{aligned} 2P_n &\equiv 3n^2 - n \pmod 3 \\ &\equiv -n \pmod 3 \\ &\equiv 2n \pmod 3, \end{aligned}$$

so $P_n \equiv n \pmod 3$.

# Borwein's Conjecture (the + - - conjecture)

Given

$$H_n(q) = \prod_{i=1}^{n}(1 - q^{3i-2})(1 - q^{3i-1}) = \sum_{j=0}^{\infty} a(j)q^j, \qquad (1)$$

## Conjecture

*If $a(j)$ is defined by (1), then for all $j \geq 0$,*

$$a(3j) \geq 0, \qquad a(3j+1) \leq 0, \qquad a(3j+2) \leq 0. \qquad (2)$$

# Partition Interpretation

We may consider $H_n(q)$ as a generating function for partitions into distinct parts, none of which are congruent to 0 mod 3, which subtracts the number of such partitions with an odd number of parts from those with an even number of parts.

## Example

| 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|
| 5 1 | 7 | 8 | 8 1 | 10 | 11 |
| 4 2 | 5 2 | 7 1 | 7 2 | 8 2 | 10 1 |
| | 4 2 1 | 5 2 1 | 5 4 | 7 2 1 | 8 2 1 |
| | | | | 5 4 1 | 7 4 |
| | | | | | 5 4 2 |
| $+2$ | $-1$ | $-1$ | $+3$ | $-2$ | $-1$ |

# Borwein-Like Theorem

## Theorem

*Defining the sequence* $b(n)$ *by*

$$F(q) = \sum_{n=0}^{\infty} b(n) q^n = \prod_{k=1}^{\infty} \frac{(1-q^{3k-1})(1-q^{3k-2})}{(1+q^{3k-1})(1+q^{3k-2})}$$

*we will have* $b(3j+2) = 0$ *for all integers* $j \geq 0$*.*

# Borwein-Like Conjecture

**Conjecture $(+-0)$**

*In the series* $F(q) = \sum_{n=0}^{\infty} b(n)q^n$, *we will have*

$$b(3n) \geq 0, \qquad b(3n+1) \leq 0, \qquad \textit{and} \qquad b(3n+2) = 0$$

*for all* $n \geq 0$.

# Borwein-Like Theorem

## Theorem (equivalent form)

*Let* $n \equiv 2 \pmod 3$. *Consider overpartitions of* $n$ *into parts not divisible by* 3. *Then the number of such overpartitions into an even number of parts equals the number of such overpartitions into an odd number of parts.*

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## Example ($n = 5$)

| | | | |
|---|---|---|---|
| $5$ | $22\bar{1}$ | $41$ | $2111$ |
| $\bar{5}$ | $\bar{2}2\bar{1}$ | $\bar{4}1$ | $\bar{2}111$ |
| $221$ | $11111$ | $4\bar{1}$ | $2\bar{1}11$ |
| $\bar{2}21$ | $\bar{1}1111$ | $\bar{4}\bar{1}$ | $\bar{2}\bar{1}11$ |

# Proof: Jacobi Triple Product

## Theorem (Jacobi Triple Product)

$$(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty = \sum_{n\in\mathbb{Z}} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$$

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## Sketch of Proof.

We deduce via the Jacobi triple product that

$$F(q) = \frac{(q;q^3)_\infty(q^2;q^3)_\infty}{(-q;q^3)_\infty(-q^2;q^3)_\infty} = \frac{(q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty}{(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty}$$
$$= \frac{\sum_{n\in\mathbb{Z}}(-1)^n q^{P_n}}{\sum_{n\in\mathbb{Z}} q^{P_n}},$$

where $P_n = \frac{1}{2}n(3n-1)$ is the $n$th pentagonal number.

# Proof: Sieve

## Sketch of Proof, cont.

We sieve $F(q)$ to get $G(q) = \sum_{n=0}^{\infty} b(3n+2)q^{3n+2}$ as follows.
Let $\omega$ be a primitive 3rd root of unity. Then

$$S(q) = F(q) + \omega F(\omega q) + \omega^2 F(\omega^2 q)$$

# Proof: Sieve

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Let $\omega$ be a primitive 3rd root of unity. Then

$$\begin{aligned} S(q) =& F(q) + \omega F(\omega q) + \omega^2 F(\omega^2 q) \\ =& b(0) + b(1)q + b(2)q^2 + b(3)q^3 + b(4)q^4 + b(5)q^5 + \cdots \\ &+ \omega b(0) + \omega^2 b(1)q + b(2)q^2 + \omega b(3)q^3 + \omega^2 b(4)q^4 + b(5)q^5 + \cdots \\ &+ \omega^2 b(0) + \omega b(1)q + b(2)q^2 + \omega^2 b(3)q^3 + \omega b(4)q^4 + b(5)q^5 + \cdots \end{aligned}$$

# Proof: Sieve

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Let $\omega$ be a primitive 3rd root of unity. Then

$$\begin{aligned}
S(q) =& F(q) + \omega F(\omega q) + \omega^2 F(\omega^2 q) \\
=& b(0) + b(1)q + b(2)q^2 + b(3)q^3 + b(4)q^4 + b(5)q^5 + \cdots \\
&+ \omega b(0) + \omega^2 b(1)q + b(2)q^2 + \omega b(3)q^3 + \omega^2 b(4)q^4 + b(5)q^5 + \cdots \\
&+ \omega^2 b(0) + \omega b(1)q + b(2)q^2 + \omega^2 b(3)q^3 + \omega b(4)q^4 + b(5)q^5 + \cdots \\
=& 0b(0) + 0b(1)q + 3b(2)q^2 + 0b(3)q^3 + 0b(4)q^r + 3b(5)q^5 + \cdots
\end{aligned}$$

# Proof: Sieve

## Sketch of Proof, cont.

We sieve $F(q)$ to get $G(q) = \sum_{n=0}^{\infty} b(3n+2)q^{3n+2}$ as follows.
Let $\omega$ be a primitive 3rd root of unity. Then

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&+ \omega^2 b(0) + \omega b(1)q + b(2)q^2 + \omega^2 b(3)q^3 + \omega b(4)q^4 + b(5)q^5 + \cdots \\
=& 0b(0) + 0b(1)q + 3b(2)q^2 + 0b(3)q^3 + 0b(4)q^r + 3b(5)q^5 + \cdots \\
=& 3G(q).
\end{aligned}$$

# Proof: Denominator is Irrelevant

## Sketch of Proof, cont.

Writing

$$F(q) = \frac{\sum_{n\in\mathbb{Z}}(-1)^n q^{P_n}}{\sum_{n\in\mathbb{Z}} q^{P_n}} = \frac{A(q)}{B(q)},$$

we get

$$G(q) = \frac{A(q)B(\omega q)B(\omega^2 q) + \omega A(\omega q)B(\omega^2 q)B(q) + \omega^2 A(\omega^2 q)B(q)B(\omega q)}{3B(q)B(\omega q)B(\omega^2 q)}.$$

# Proof: Denominator is Irrelevant

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we get

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We see that the numerator is a $q^{3n+2}$ sieve of $3A(q)B(\omega q)B(\omega^2 q)$, so the coefficients of $q^{3n}$ and $q^{3n+1}$ will be 0. If we can show that the numerator is in fact zero, then $G(q)$ will be zero as well.

# Proof: Closed Form of Coefficients

## Sketch of Proof, cont.

We find that the numerator is equal to $\sum_{N=0}^{\infty} R(N)q^N$, where

$$R(N) = \sum_{\substack{k,m,n\in\mathbb{Z} \\ P_k+P_m+P_n=N}} \omega^{m-n}\left[(-1)^k + (-1)^m\omega + (-1)^n\omega^2\right].$$

We will define $f(k,m,n) = \omega^{m-n}\left[(-1)^k + (-1)^m\omega + (-1)^n\omega^2\right]$ so that we may write

$$R(N) = \sum_{\substack{k,m,n\in\mathbb{Z} \\ P_k+P_m+P_n=N}} f(k,m,n).$$

# Proof: Defining a Function $\phi$

## Sketch of Proof, cont.

Letting $X = \{(k, m, n) \in \mathbb{Z}^3 \mid k + m + n \equiv 2 \pmod 3\}$, we define a function $\phi : X \longrightarrow X$ by

$$\phi(k, m, n) = \left(k - \frac{2s-1}{3}, m - \frac{2s-1}{3}, n - \frac{2s-1}{3}\right) = (k', m', n'),$$

where $s = k + m + n$. This is a well-defined function, since $2s - 1 \equiv 2 \cdot 2 - 1 \equiv 0 \pmod 3$ and

$$s' = k' + m' + n' = s - (2s - 1) = 1 - s \equiv 2 \pmod 3.$$

We will also have $P_k + P_m + P_n = P_{k'} + P_{m'} + P_{n'}$, although we will not show this here.

# Proof: $\phi$ is an Involution

## Sketch of Proof, cont.

We have

$$\phi(k,m,n) = \left(k - \frac{2s-1}{3}, m - \frac{2s-1}{3}, n - \frac{2s-1}{3}\right) = (k', m', n')$$

where $s = k + m + n$. We also see that $\phi$ is an involution, since

$$\begin{aligned}\phi(k', m', n') &= \left(k' - \frac{2s'-1}{3}, \cdots\right)\\ &= \left(k - \frac{2s-1}{3} - \frac{2(1-s)-1}{3}, \cdots\right)\\ &= (k,m,n),\end{aligned}$$

with the second and third terms analogous to the first.

# Proof: $\phi$ Changes Sign of $f$

## Sketch of Proof, cont.

We see that

$$\frac{2s-1}{3} \equiv 2s-1 \equiv 1 \pmod{2},$$

and

$$m' - n' = \left(m - \frac{2s-1}{3}\right) - \left(n - \frac{2s-1}{3}\right) = m - n,$$

so

$$\begin{aligned} f(\phi(k,m,n)) &= \omega^{m'-n'}\left[(-1)^{k'} + (-1)^{m'}\omega + (-1)^{n'}\omega^2\right] \\ &= -\omega^{m-n}\left[(-1)^{k} + (-1)^{m}\omega + (-1)^{n}\omega^2\right] \\ &= -f(k,m,n). \end{aligned}$$

# Proof: Conclusion

## Sketch of Proof, cont.

Finally, we see that

$$\begin{aligned} R(N) &= \sum f(k,m,n) \\ &= \frac{1}{2}\left[\sum f(k,m,n) + \sum f(\phi(k,m,n))\right] \\ &= \frac{1}{2}\left[\sum f(k,m,n) + \sum [-f(k,m,n)]\right] \\ &= 0, \end{aligned}$$

as desired. □

# General Theorem

## Theorem (Overpartition Nonresidue)

*Given a prime $p$, define the sequence $b_p(n)$ by*

$$F_p(q) = \sum_{n=0}^{\infty} b_p(n) q^n = \prod_{k=1}^{\infty} \frac{(1-q^k)(1+q^{kp})}{(1+q^k)(1-q^{kp})} = \frac{(q;q)_\infty(-q^p;q^p)_\infty}{(-q;q)_\infty(q^p;q^p)_\infty}.$$

*Then when $n$ is a quadratic nonresidue modulo $p$, we have*

$$b_p(n) = 0.$$

# Additional Conjectures

**Theorem (Overpartition Nonresidue, equivalent form)**

*Let $n$ be a quadratic nonresidue modulo a prime $p$. Consider the overpartitions of $n$ into parts not divisible by $p$. Then the number of such overpartitions into an even number of parts equals the number of such overpartitions into an odd number of parts.*

# Proof (Andrews): An Identity

Sketch of Proof.

We have the identity

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \frac{(q;q)_\infty (q;q)_\infty}{(-q;q)_\infty (q;q)_\infty} = (q;q^2)_\infty (q;q)_\infty$$
$$= (q;q^2)_\infty^2 (q^2;q^2)_\infty = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}$$

by the Jacobi triple product.

# Proof (Andrews): Expression for $b_p(n)$

## Sketch of Proof, cont.

We may rewrite $F_p(q)$ as

$$\sum_{n=0}^{\infty} b_p(n)q^n = \frac{(q;q)_\infty(-q^p;q^p)_\infty}{(-q;q)_\infty(q^p;q^p)_\infty}$$

$$= \left(\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\right)\left(\sum_{n=0}^{\infty} g(n)q^{pn}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{\substack{-\lfloor\sqrt{n}\rfloor\le k\le\lfloor\sqrt{n}\rfloor \\ p\mid n-k^2}}(-1)^k g\left(\frac{n-k^2}{p}\right)\right)q^n.$$

# Proof (Andrews): Conclusion

Sketch of Proof, cont.

Thus

$$b_p(n) = \sum_{\substack{-\lfloor\sqrt{n}\rfloor \le k \le \lfloor\sqrt{n}\rfloor \\ p \mid n-k^2}} (-1)^k g\left(\frac{n-k^2}{p}\right).$$

When $n$ is a quadratic nonresidue modulo $p$, there will be no $k$ so that $p \mid n - k^2$, so the sum will be empty and $b_p(n) = 0$. □

# References


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- "Overpartitions", S. Corteel and J. Lovejoy, 2004.
- "Partition Theorems in Need of Bijections", K. Garrett, 2003.