m-gapped Progressions and van der Waerden Numbers

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Background

Definition

An r-coloring on a set A is a function $\mu : A \to \{1, 2, ...r\}$.
Equivalently, an r-coloring is a partition of A into r different sets.
We say that elements $a_1, a_2...a_k \in A$ are monochromatic if
$\mu(a_1) = \mu(a_2) = ... = \mu(a_k)$

van der Waerden's theorem

For any $k, r \in \mathbb{N}$, there exists an integer N such that any r-coloring of $[1, N]$ contains a monochromatic k-term arithmetic progression.

Definition

For $k, r \in \mathbb{N}$, the van der Waerden number $W(k; r)$ is defined as the smallest integer N for which the above statement is true.

m-gapped progressions

- In 2015, the REU group led by Bruce Landman began studying how van der Waerden numbers could be generalized from arithmetic progressions to the more general m-gapped progressions.

Definition

An m-gapped progression is a sequence of ordered integers $(a_1, a_2 ... a_k)$ such that $|\{a_{i+1} - a_i | 1 \leq i \leq k-1\}| \leq m$

- Ex: The sequence $\{1, 3, 6, 9, 11\}$ is a 2-gapped progression, because the set of gaps between consecutive elements is $\{2, 3\}$.
- Under this definition, arithmetic progressions are equivalent to 1-gapped progressions.

Defining $B_m(k; r)$

Definition

Let $m, k, r \in \mathbb{N}$. We define $B_m(k; r)$ as the smallest positive integer N such that any r-coloring of $[1, N]$ contains a monochromatic k-term m-gapped progression.

- We will often write $B_m(k)$ to refer to $B_m(k; 2)$.
- For any fixed $k, r \in \mathbb{N}$, we have that $W(k; r) = B_1(k; r) \geq B_2(k; r) \geq B_3(k; r) \dots$
- The above fact, along with van der Waerden's theorem, implies that $B_m(k; r)$ exists for any m, k, r.

Upper Bounds for $B_m(k)$

- We know that for any m, $B_m(k) \leq W(k)$, so this immediatley gives us an upper bound.
- Currently, the best known upper bound for $W(k)$ is $W(k) \leq 2^{2^{2^{2^{2^{k+9}}}}}$

Theorem

Let m be odd. Then $B_m(k) \leq mW(\lceil \frac{2k}{m+1} \rceil)$

- While still large, this upper bound is likely significantly smaller than the upper bound of just the van der Waerden numbers.
- Ex: To get an upper bound for $B_3(6)$, our bound gives $B_3(6) \leq 3W(3) = 27$, while $W(6) = 1132$.

Outline of proof

- Let $W = W(\lceil \frac{2k}{m+1} \rceil)$. Take any coloring of $[1, mW]$.
- Consider the colorings of the subintervals $[1, W], [W+1, 2W]...[(m-1)W+1, mW]$
- Each subinterval must contain a red or blue arithmetic progression of length $\lceil \frac{2k}{m+1} \rceil$
- We can then combine either enough red arithmetic progressions, or enough blue arithmetic progressions, to get a progression of the desired length that contains at most m gaps.

Values of $B_m(k)$

Table: $B_m(k;2)$

k\m	2	3	4	5
3	5	5	5	5
4	9	7	7	7
5	14	9	9	9
6	21	11	11	11
7	28	15	13	13
8	41	19	15	15
9		23	17	17
10		27	19	19
11		32	23	21
12			25	23
13			29	25
14			31	27
15			35	29
16				33
17				35
18				37

- From values, notice that entries starting any column are given by $B_m(k) = 2k - 1$
- After this, next value is $2k + 1$.
- Notation: $T_m = \frac{m(m+1)}{2}$

Proposition 1

Let $k \leq T_m$. Then, $B_m(k) = 2k - 1$.

Proposition 2

Let $m \geq 2$, and $k = T_m + 1$. Then, $B_m(k) = 2k + 1$.

Idea of Proposition 1:

- Know that $B_m(k) \geq 2k - 1$, since $[1, 2k - 2]$ can be 2-colored so that neither color has k elements
- Any 2-coloring of $[1, 2k - 1]$ must contain either k red or k blue elements.
- Can show by contradiction that if the k elements of the same color contain more than m gaps between consecutive elements, then the sum of all gaps in that color is greater than $2k - 2$.

$B_m(k)$ for $k > T_m$

- Let $k = T_m + 1$. Then, a coloring of $[1, 2k]$ containing no monochromatic k term m-gapped progressions is: $RBR^mBBR^{m-1}BBBR^{m-2}...B^{m-1}RRB^mRB$

Conjecture

Let $m \geq 2$, and $k = T_m + 1$. Then, up to recoloring all elements, there are $(m!)^2$ colorings of $[1, 2k]$ that do not contain monochromatic k term m-gapped progressions.

- By extending this coloring, we get that when $k \geq T_m + 1, B_m(k) \geq 2k + 1$.
- Have shown that when $k = T_m + s$, where $1 \leq s \leq \frac{m-1}{2}$, then $B_m(k) = 2k + 1$.

$B_m(k;r)$ for general r

Table: $B_m(k;3)$

k \m	2	3	4	5	6
3	7	7	7	7	7
4	16	10	10	10	10
5		17	13	13	13
6			19	16	16
7				19	19
8				22	22
9				28?	25?

Proposition

Let $k \leq \frac{m(m+1)+2r-4}{2r-2}$. Then, $B_m(k;r) = rk - r + 1$.

Conjecture

Let $m \geq 2r-2$, and $k = \lfloor \frac{m(m+1)+2r-4}{2r-2} \rfloor + 1$. Then, $B_m(k;r) = kr + 1$

So far, we believed we have shown that $B_m(k;r) \leq kr + 1$ in the above case.

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Thank you