1 Introduction

1.1 Random Triangles

Within the realm of geometric probability, the idea of a random triangle has been around for at least 120 years. One of the best early examples of a random triangle in published literature comes from author and mathematician Charles Dodgson, perhaps better known under his pen name Lewis Carroll. In 1893, Dodgson published *Pillow Problems*, a collection of 72 original math problems that were challenging enough to keep him occupied late into the night while dealing with bouts of insomnia. One of these proposed questions became the foundation for our exploration into random triangles, and it can be stated like this: What is the probability that three points in the plane chosen randomly (uniformly) will form an acute triangle?

1.2 Dodgson Proof

The solution that Charles Dodgson gave was clever, but ultimately incorrect. His proof relies heavily on Figure 1 below, and begins with an important and fairly natural assumption. Every triangle will have three sides, and one of these sides will (with probability 1) be the longest side. Suppose that we have the longest side. By only considering the possible locations of the third point, conditioned on the first two points forming the longest side, Dodgson was able to arrive at a clever solution.

In Figure 1, the line segment forming the hypotenuse of the right triangle is the longest side that Dodgson conditioned on. Let’s call the length of this side $D$. The two circles of radius $D$ centered at the endpoints of the segment represent all of the possible locations of the third point such that the distance from this new point to the endpoint is smaller than $D$. The overlap of these two circles, then, becomes the possible locations of the third point such that
the assumption that the initial side of length \( D \) is the longest side is true. The area of this lune shaped region can be found using known formulas for the area of overlapping circles, specifically noting that in our case, the distance between the two centers is equal to the radius of each circle. Thus:

\[
A_{\text{lune}} = 2 \times D^2 \cos^{-1}\left(\frac{1}{2}\right) - \frac{\sqrt{3}}{2} \times D^2
\]

Now, to find the locations of the third point that would result in an acute triangle, Dodgson considered the circle inscribed in the lune-shaped region with diameter \( D \). Any point that falls on the circumference of this circle will form a right triangle, which can be confirmed with some basic trigonometry. Any point that falls within the circle, then, will form an obtuse angle. This means that the gray region in the graphic above is the region that will result in an acute triangle. Since: \( A_{\text{circle}} = \pi \times \left(\frac{D}{2}\right)^2 \),

\[
A_{\text{gray}} = A_{\text{lune}} - A_{\text{circle}}
\]

and

\[
P(\text{Acute}) = \frac{A_{\text{gray}}}{A_{\text{lune}}} = \frac{\pi/8}{\frac{\pi}{3} - \frac{\sqrt{3}}{4}} \approx 0.64
\]

### 1.3 Problems with Dodgson’s Proof

The clever approach that Dodgson used is appealing, but unfortunately it is not an accurate proof, at least not for the question that he stated. While
Dodgson’s result was known to have problems for awhile, a nice counter-proof was derived and published in 1973 by Frank Wattenberg. This proof relies on assuming our distance $D$ is now the second longest line. With this assumption, the ratio of appropriate areas becomes:

$$P(\text{Acute}) = \frac{\pi}{\frac{\pi}{3} + \frac{\sqrt{3}}{2}} \approx 0.82$$

Because both proofs seem equally valid (or perhaps invalid), it is obvious that something is wrong with the conditioning. By taking the ratio of areas, the proof assumes that the 3rd point has a uniform distribution on the plane. However, the distribution of the 3rd point given that the other two points form the longest side is evidently non-uniform.

The idea of selecting points randomly in the plane has some issues. There is no way of having a uniform density on all of $\mathbb{R}$. Instead, you must reduce the plane to some compact subset, or choose a non-uniform density with infinite support. The most natural choice for a distribution with infinite support is the normal distribution.

### 1.4 Multivariate Normal Distribution

A multivariate normal distribution is just a way of extending the one-dimensional normal distribution to $\mathbb{R}^n$. A point drawn from an n-dimensional normal distribution simply has each of its n coordinates drawn from an independent, identically distributed normal distribution. For our purposes, we always choose the standard normal distribution. The resulting multivariate normal is sometimes referred to as a Gaussian ball.

Probabilists have studied the pillow problem using a bivariate normal distribution instead of Dodgson’s assumed uniform distribution. Steven Portnoy provided a result in his 1994 paper *A Lewis Carroll Pillow Problem: Probability of an Obtuse Triangle*. He showed that, in two dimensions:

$$P(\text{Acute}) = \frac{1}{4}$$

A similar question to the one Portnoy answered is that of a pinned Gaussian ball. In this set up, instead of selecting all three points randomly, you fix the first point at the origin (also the mean of the distribution) and select just two points randomly. In this case,

$$P(\text{Acute}) \approx 0.208$$
2 Uniform Case

2.1 Definitions

The problem we became interested in was the probability of acute triangles on the sphere. Before directly exploring the problem, it is necessary to be precise about specific terms that will be continuously used throughout this paper.

1. Line: A line, which is formed on the surface of the sphere is an arc between two points. It is the shortest distance between the two points and in order to meet this condition it must be on the trajectory of the great circle.

2. Length of the arc: The length of the arc is calculated using the formula of the interior angle multiplied by the radius of the sphere. The interior angle is the angle that is formed in the center of the sphere by the two points. Because the probability as a ratio is independent of the radius of the sphere, we naturally chose the simplest sphere: the unit sphere. Since $r = 1$, the arc length directly reflects the value of the interior angle.

3. Acute triangle: In an acute spherical triangle, all three angles of the spherical triangle have to be less than $\pi/2$. Observe that the length of each side has to be less than $\pi/2$.

2.2 Spherical Law of Cosines

One theorem that is necessary to solve this problem is the spherical law of cosines, which can be stated as:
\[ \cos c = \cos a \times \cos b + \sin a \times \sin b \times \cos C \]

Refer to Figure 2, where \(a, b,\) and \(c\) are the side lengths, and \(C\) is the angle opposite side \(c\).

However for our purposes it is sufficient to only consider the case in which \(C = \pi/2\). If \(C = \pi/2\), in this case we have a relationship between the 3 sides of the spherical triangle to be:

\[ \cos c = \cos a \times \cos b \]

2.3 Methodology

We start out by defining a triangle by consecutively picking the 3 points. In the uniform distribution, the first point can be anywhere so without loss of generality we define the first point to be have a Cartesian coordinate of (1,0,0). Next, we pick the second point and denote the length of the line formed by the first point and second point \(D\). Since we are interested in points forming an acute triangle, \(D\) has to have a length shorter than \(\pi/2\). Thus, given that the first point is fixed at (1,0,0), the possible area that the second point can be is the half sphere. Although we are working with a uniform distribution on the sphere, the \(P(D = \theta)\) for \(0 \leq \theta \leq \pi/2\) is not a uniform distribution. Thus to accommodate this, a function for the density of the distance, \(f_D\), is necessary.

2.4 Density of Distance

First, we will try to build the cumulative distribution function, \(F_D(x)\). To do this, we consider the probability that our random variable \(D\) is less than some arc length, \(\theta\). Then, we turn our attention to Figure 3.

There are a whole bunch of points at any distance \(x\) away from our first point (at the top of the sphere pictured above). These points will form a circle of radius \(r = \sin(x)\). To find \(P(D < \theta)\), we simply add up all of these circles from 0 to \(\theta\). That is:

\[
F_D(\theta) = P(D < \theta) = \int_0^\theta 2\pi \sin(x) \frac{1}{4\pi} dx
\]

To find the density of \(D\), we just differentiate \(F_D\) to arrive at:

\[
f_D(\theta) = \frac{\sin(\theta)}{2}
\]
2.5 Location of the Third Point

After fixing the first point to be at (1, 0, 0) and found the density for the distance between the first point and the second point. The next step is to find out: given the first two points fixed, and the third point is chosen randomly, what is the probability that the three points can form an acute triangle? To solve for this probability, we need to find out what region on the sphere can the third point fall into such that they form an acute triangle.

As shown in Figure 4, we let the angles at the first two points be base angles, and the angle at the third point be the top angle. In order for the two base angles to be acute, the third point must be bounded by the pizza region. Figure 4 is only showing the top hemisphere, the bottom is symmetric to the top half. In order for the top angle to be acute, the third point cannot fall into the region bounded by the locus of points that can form a right angle with the first two points. Therefore, given the first two points fixed, the third point has to be within the shaded region.

2.6 Spherical Coordinates

Knowing exactly what region the third point should fall into to form an acute triangle, we need to find a way to calculate the area of the shaded region. Since we are examining an area on a sphere here, it would be most natural
Using spherical coordinate, any point $v$ within the region can be expressed in terms of two coordinates, $(\phi, \theta)$. $\phi$ is the angle between $v$ and the $z$ axis. Since $v$ is on the unit sphere, $\phi$ is also the distance from $(0, 0, 1)$ to $v$. Similarly, $\theta$ is the angle between the $(1, 0, 0)$ and the projection of $v$ on the $x, y$ plane. $\theta$ is also the distance from $(1, 0, 0)$ to the intersection of the line joining $(0, 0, 1), v$ and the equator as shown in the figure above. Remember from calculus that integrating over an area in spherical coordinates requires a Jacobian transformation, namely:

$$dA = r^2 \sin(\phi) d\phi d\theta$$
Once we established the proper coordinate system, we are ready to integrate over the region.

### 2.7 Integration

The first step is to express the length of the vertical lines that are coming down from the north pole to the equator in terms of $\theta$. To do that we calculate the length $L$, which is the length of the vertical lines that are beneath the arc. Once we have the value of length $L$, the length of the vertical lines can be calculated by $\pi/2 - L$. Calculation of $L$ can be done by using the spherical law of cosines, as shown in Figure 6.

We find:

$$L(\theta) = \cos^{-1}\left(\frac{\cos(D)}{\cos(\theta) \times \cos(D - \theta)}\right)$$

From Figure 6 we can see that because the arc is the locus of all points that has a right angle at the third point, we can use the spherical law of cosines to establish a relationship of

$$\cos(D) = \cos(x) \times \cos(y)$$

Additionally because all vertical lines from the north pole form a right angle
with the equator,

\[ \cos(x) = \cos(L) \times \cos(\theta) \]

\[ \cos(y) = \cos(L) \times \cos(D - \theta) \]

Thus

\[ \cos(D) = \cos(L) \times \cos(\theta) \times \cos(L) \times \cos(D - \theta) \]

\[ \cos(L) = \sqrt{\frac{\cos(D)}{\cos(\theta) \times \cos(D - \theta)}} \]

\[ L(\theta) = \cos^{-1}\left(\sqrt{\frac{\cos(D)}{\cos(\theta) \times \cos(D - \theta)}}\right) \]

Once we have an expression of the length \( L \) in terms of \( \theta \), the sum of all vertical lines for all \( \theta \) will calculate the shaded area from Figure 4. Thus we have the integral which calculates the area of acute triangles for a given \( D \).

\[
\int_{\theta=0}^{D} \int_{\phi=0}^{\frac{\pi}{2} - L(\theta)} \frac{1}{4\pi} \sin(\phi) \, d\phi \, d\theta
\]

Finally, as the distance between the first point and the second point can vary between 0 and \( \pi/2 \) so an integral of \( D = 0 \) to \( D = \pi/2 \) with the density function of the length \( D \) has to be added. Thus the final integration is

\[
P(\text{Acute}) = 2 \int_{D=0}^{\frac{\pi}{2}} \int_{\theta=0}^{D} \int_{\phi=0}^{\frac{\pi}{2} - L(\theta)} \frac{1}{4\pi} \sin(\phi) \, d\phi \, d\theta \, f_D \, dD
\]

The two multiplied in front of the integration is from the symmetry of the upper hemisphere and the lower hemisphere.

2.8 Integration Results

Since a closed form formula for the integral could not be derived, we decided to numerically estimate the integral. We broke down the domain of integration into cubes of side length 0.01 and calculated the density value for each of the cubes. At last we multiplied the density with the volume of the cube and summed up all the values. It was calculated to be 0.034.
To verify the results were valid, a real simulation of the problem was done. We did it by drawing three points at random on a uniform sphere. We then calculated each angle of the triangle and checked if all of them are acute. Finally we calculated the proportion such triangles to be 0.034, which agreed with our integral estimation.

3 Non-uniform Case

3.1 Von Mises-Fisher Distribution

While multivariate wrapped normal distributions exist, their densities are rather unwieldy. A more natural analog for the normal distribution on the sphere is the von Mises-Fisher distribution. The von Mises-Fisher distribution arose from the field of directional statistics in the earth 20th century. It is useful when all you know (or care) about some data set is direction, but not amplitude. Some examples include the trajectory of homing pigeons as they set out on their route, or the direction of magnetic poles in ancient rocks. The von Mises distribution was originally only on $S^1$, and the Fisher distribution arose as an extension to $S^2$. The entire family of distributions depends on two parameters, the mean direction $\mu$ and the spread parameter $\kappa$. The densities are as follows:

$$f_p(x|\mu, \kappa) = C_p(\kappa)e^{\kappa \mu^T x}$$

where $C_p(\kappa)$ is the normalizing constant and $p$ is the dimension of the hypersphere. Since $\mu^T x = \cos(\theta)$, where $\theta$ is the distance between $x$ and $\mu$ on the unit hypersphere, we can rewrite this density as a function of $\theta$:

$$f_p(\theta|\kappa) = C_p(\kappa)e^{\kappa \cos(\theta)}$$

With some basic knowledge of the von Mises-Fisher distribution, we are ready to tackle the question of spherical triangles again. The caveat this time is that we are going to set the first point in our triangle to be at $\mu$, the mean direction, and then select two more points via some von Mises-Fisher density. This is much like the pinned Gaussian ball question on the plane, as the von Mises-Fisher distribution can be seen as a spherical analogue to the bivariate normal.

3.2 Density of Distance

Similar to the uniform case, we first concern ourselves with understanding the distance between any two random points, and consider this random variable
Much like in the uniform case, we consider the circle of points an arc length $x$ away from our first point, $\mu$. The difference in this case is that instead of every circle having the uniform density, each will have a density defined by its distance from $\mu$. Thus:

$$F_D(\theta) = P(D < \theta) = \int_0^{\theta} 2\pi \sin(x) f(x) \, dx$$

Differentiating, we find:

$$f_D(\theta) = 2\pi \sin(x) f(x)$$

Because the uniform density of a point is $\frac{1}{4\pi}$, it can be noted that this is actually just a generalization of the uniform case.

### 3.3 Pizza Region: Revisited

Once we have $f_D$, we once again arrive at the same pizza region as in the uniform case. The important difference to note is that, because we set the first point to be at the mean direction, $\mu$, the bottom corner of the region is actually $\mu$. 
Unlike in the uniform case, we have to be concerned with the distance, $A$, that any given point is from $\mu$. From Figure 8, applying the spherical law of cosines, we can create a function for $A$ in terms of $\theta$ and $l$, where $l = \frac{\pi}{2} - \phi$. Then:

$$A(\theta, l) = \cos^{-1}(\cos(\theta) \times \cos(l))$$

Recall from the uniform case our expression for $L(\theta)$, the height of the region of points that form an obtuse angle:

$$L(\theta) = \cos^{-1}\left(\sqrt{\frac{\cos(D)}{\cos(\theta) \times \cos(D - \theta)}}\right)$$

We can now use similar methods to derive our final integral.

$$P(\text{Acute}) = 2 \int_{0}^{\pi/2} \int_{0}^{D} \int_{0}^{\pi/2-L(\theta)} f(A(\theta, l)) \sin(\phi) \, d\phi \, d\theta \, f_D \, dD$$

where $f$ is the von Mises-Fisher density written in terms of distance from $\mu$, and $f_D$ is the density of the distance. However, since our density is actually a family of densities dependent on the spread parameter $\kappa$, we did not produce just one solution.

We also believe that our methodology can easily be extended to any and all distributions on the sphere that have the property that all points a distance $\theta$ from the mean have the same density.
3.4 Integration Results

Using the same technique of discretization, we are able to derive probabilities for a variety of spread parameter $\kappa$ values. A graph of probability vs. $\kappa$ is shown below:

To validate our results, we also did a simulation of the real problem and plotted their results below:

By superimposing the two results we can obtain the following graph:
Although numerical integration and the simulations are both estimations of the real probabilities, they stayed close enough for us to believe our results were valid.

3.5 Observations on the Probability Curve

When $\kappa \to 0$

As we can observe from the graphs in the previous section, when $\kappa$ approaches 0, the probability is approximately 0.034, the probability that we obtained in a uniform case. This tells us that the uniform distribution on a sphere is just a extreme case of the von Mises-Fisher distribution.

When $\kappa \to \infty$

The curve is sloping upwards, and it is approaching some kind of the limit at the right end. The limit was around 0.2, and we believe that the limit is the probability of finding an acute triangle in a pinned Gaussian ball, mentioned in section 1.4.

The intuition is that when the von Mises-Fisher distribution is concentrated at the mean direction, and locally it resembles a bivariate normal distribution. Therefore, the probabilities will also resemble those of the bivariate Normal distribution.
To prove the above conjecture more rigorously, we first take a look at the original von Mises-Fisher density:

$$f_p(x|\mu, \kappa) = C_p(\kappa) e^{\kappa \mu^T x}$$

If we let the angle between $x$ and $u$ to be $\theta$, then the density of $x$ becomes:

$$f_p(x|\mu, \kappa) = C_p(\kappa) e^{\kappa \cos(\theta)}$$

When $\kappa \to \infty$, the distribution is extremely concentrated around the mean direction, therefore, $\theta \to 0$. Since for small $\theta$, $\cos(\theta) \approx 1 - \theta^2 / 2$, we can express the density as

$$f_p(x|\mu, \kappa) \approx Ae^{-\kappa \theta^2 / 2}$$

for some normalizing constant $A$. Since $x$ is on the unit sphere, $\theta$ is also the distance from the mean direction to vector $x$. Notice that the above formula is extremely similar to the density formula for the normal distribution.

Recall that the density formula for any point $(x, y)$ on a plane of a bivariate normal distribution is

$$f_{bn}(x, y) = Ae^{-\kappa(x^2 + y^2) / 2}$$

If we let $\theta$ be the distance of from point $(x, y)$ to the origin, then the density formula becomes:

$$f_{bn}(x, y) = Ae^{-\kappa(\theta^2) / 2}$$

which is exactly equal to the density approximation of the von Mises-Fisher distribution when $\kappa$ is very large. This shows that the limit that the probability curve is approaching is indeed the probability of finding an acute triangle in a pinned Gaussian ball.

References

[1] M. Yoichi, *Computational Geometry and Graph Theory: International Conference, KyotoCGGT, Seven Types of Random Spherical Triangles in $S^n$ and Their Probabilities.* (Springer Berlin Heidelberg)