The Dynamics of Quadratic Mappings

Dynamical Systems is the study of the orbit structure of mappings from a set to itself. If \( f : S \to S \) and \( x \in S \), then the orbit of \( x \) is \( O(x) = \{ x, f(x), f(f(x)), \ldots \} \). Examples of dynamical questions include “Does the orbit contain finitely many distinct points?” and “What is the asymptotic behavior of “most” orbits?”

As an application, if \( S \) be the state space for some physical or biological system then \( f \) is the rule that describes how the system evolves from one state to the next and the long-term behavior of the system is described by the asymptotic properties of its orbits.

A dynamical system is said to be \textit{chaotic} if the sequence of points in an orbit appear to be random in spite of the fact that \( f \) stipulates the evolution quite deterministically. The discovery and study of chaos within deterministic systems has been an active area in science and mathematics for the past few decades.

It turns out that, although often difficult to analyze, chaotic dynamical systems are not rare. A necessary condition for chaos is that \( f \) be non-linear and the simplest non-linear functions are quadratic ones. Here are two chaotic families of dynamical systems - the first on the real line and the second on the plane.

\[
Q_c(x) = c - x^2 \quad c \in \left( \sqrt{2}, 2 \right)
\]

and

\[
H_{a,b}(x, y) = \left( a - by - x^2, x \right) \quad (a, b) \text{ near } (1.3, -0.3)
\]

Both \( Q \) and \( H \) are quadratic but \( Q \) is non-invertible while \( H \) is invertible. It happens that one-dimensional chaotic systems must be non-invertible. The function \( H \) is called the Hénon map and is perhaps the simplest example of an invertible chaotic dynamical system. Its dynamics are well studied but not still entirely understood.

In this project we will explore the dynamics of quadratic mappings in the plane and, especially, in three and higher dimensions. We will attempt to answer the following questions:

1. What are the chaotic quadratic dynamical systems in the plane? Are they all essentially (topologically conjugate to) the Hénon map?

2. What are the chaotic quadratic dynamical systems in \( \mathbb{R}^3 \)?

3. Is there a natural choice for a 3-dimensional Hénon map? If so, describe the dynamics.

4. It happens that \( H \) contains \( Q \) as a special case and in this case we say \( H \) \textit{unfolds} \( Q \). Which 3-dimensional quadratic maps unfold \( H \)?

5. Which of these results extend to \( \mathbb{R}^m \)?

One emphasis in this study will be on collecting, and in some cases, building sharp digital tools to analyze our dynamical systems. Another emphasis will be writing a publication-worthy report of our findings.