Symbolic Dynamics and Tilings
Fall/Winter 2017-18

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Prerequisites: Math 321 (Real Analysis) or equivalent experience

In one-dimensional symbolic dynamics, we consider all bi-infinite strings from a finite alphabet. For example, let $\mathcal{A} = \{0, 1\}$ be our finite alphabet, and let X denote the collection of all bi-infinite strings of 0's and 1's. Next, let $\sigma : X \to X$ denote the shift map defined by

$$\sigma(\ldots \omega_{-1}\omega_0\omega_1 \ldots) = \ldots \omega_0\omega_1\omega_2 \ldots$$

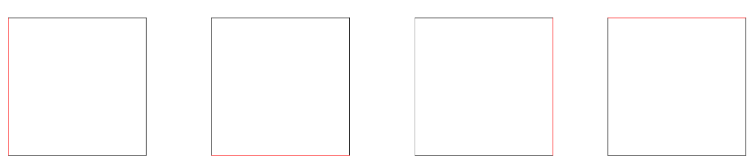
for any $\omega = \ldots \omega_{-1}\omega_0\omega_1 \ldots \in X$. Notice that any infinite string of 0's and 1's that is shifted is *still* an infinite string of 0's and 1's! We could talk about one-dimensional shift spaces all day, but the point of this comps project is to extend this idea to $\mathbb{R}^2$.

A multi-dimensional shift space is defined much in the same way, but now our 'strings' fill a larger space. For sake of clarity, let's focus on the $\mathbb{R}^2$ case. Let $\mathcal{A} = \{0, 1\}$ be our alphabet and let $X = \{0, 1\}^{\mathbb{Z}^2}$. An element of X is of the form

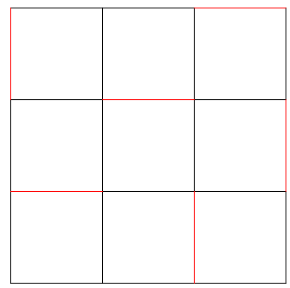
$$\begin{array}{cccccccc} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \ldots & 0 & 0 & 0 & 1 & 1 & 0 & \ldots \\ \ldots & 1 & 1 & 0 & 1 & 0 & 1 & \ldots \\ \ldots & 0 & 1 & 1 & 1 & 1 & 0 & \ldots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

As in the case of the one-dimensional shift space, we can define a shift map on the space of all these arrays of 0's an 1's. We can also look at special subsets of this space. For example, consider all elements of X that contain no adjacent 1's either horizontally or vertically. If you shift an element with this property, the resulting element still obeys the rule! In this comps project, we will study the properties of such subshifts by considering different types of restrictions. We will also explore the connections between some special subshifts and tilings in $\mathbb{R}^2$.

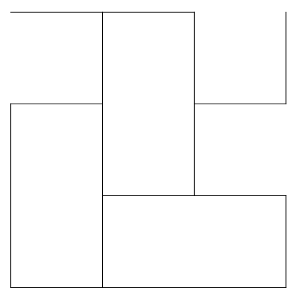
I won't go too deep into the notation for tilings, but I will give you a taste. Consider the four tiles below:



Notice that each tile has three black sides and one red side. If I only allow tiles to be adjacent if their edge color matches, then I restrict the ways in which I can arrange these tiles. Here is an example of such an arrangement:



Another fun way to think about this is to remove the red edges:



Notice that in both the case of the two-dimensional subshift and in the case of the tiling, we place some sort of restriction on the way we allowed to arrange these objects. As you may be able to see, there is a strong connection between tilings of this form and two-dimensional subshifts. We will take a closer look at these connections and explore some other questions (which will depend on the interests of the group).