The Riemann zeta-function and beyond

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Terms: Winter/Spring

Prerequisites: Math 261/361 or a commitment to self-study some complex analysis prior to winter term. (For self-study, please work through Chapters 1 – 6 of the text Fundamentals of Complex Analysis for Mathematics, Science, and Engineering by E.B. Saff and A. D. Snider.)

Description: Let s denote a complex number. The Riemann zeta-function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1.$$

Notice that when $s = 1$, the series becomes the familiar harmonic series. Euler (1730) first considered the series for s real, but it was Riemann (1859) who studied the function with s complex. By making this change Riemann showed that the function has meromorphic continuation to the complex plane, with a simple pole at $s = 1$. What does this have to do with number theory? By the Fundamental Theorem of Arithmetic, the function can also be written as the Euler product

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \Re(s) > 1.$$

It can be shown that the nontrivial zeros of $\zeta(s)$ are intimately related to the distribution of the prime numbers. The Prime Number Theorem asserts that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1, \qquad \text{where } \pi(x) = \sum_{p\leq x} 1.$$

At the heart of this proof are the zeros of the Riemann zeta-function.

Possible lines of investigation:

- **The Prime Number Theorem** – Learn the proof of the prime number theorem. This project will involve applying techniques from complex analysis to derive properties of the Riemann zeta-function, relate the prime counting function $\pi(x)$ to the Chebyshev function $\psi(x)$, and so forth.

- **The Riemann Hypothesis** – Give historical context and mathematical motivation for the statement of the Riemann Hypothesis. Report on consequences of RH. (For example, the improved error term in the Prime Number Theorem.) Report on conjectures that imply the Riemann Hypothesis. (For example, Mertens Conjecture (1885), that $|\sum_{n\leq x} \mu(n)| \leq \sqrt{x}$ for all $x > 1$. Note – this conjecture was proved to be false in 1985!). This project will involve understanding the statement of the explicit formula and analytic properties of the Riemann zeta-function.

- **The Selberg Class** – After proving the analytic properties of the Riemann zeta-function, extend this understanding to the axiomatically defined class of L-functions referred to as the Selberg Class. L-functions in this class are defined by a Dirichlet series and enjoy properties such as (1) meromorphic continuation to the entire plane, with at most a simple pole at $s = 1$, (2) the Dirichlet coefficients satisfy the Ramanajuan Conjecture, that is, their size is similar to the divisor function, (3) a functional equation, and (4) an Euler product.