Maximum Arc Digraph with a Given Zero Forcing Number

Cora Brown, Nathanael Cox

Iowa State University
Ames, IA 50011

October 29, 2013
A digraph $\Gamma = (V, E)$, is a vertex set, $V$, and an arc set of ordered pairs, $E$, where $(u, v) \in E(\Gamma)$ if $u, v \in V(\Gamma)$ and there exists an arc in $\Gamma$ that points from $u$ to $v$. 

An example of a digraph
Matrices for Digraphs

\[
\begin{bmatrix}
? & 0 & * & *\\
* & ? & * & 0 \\
0 & * & ? & 0 \\
* & * & 0 & ? \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
3 & 5 & 8 & 0 \\
0 & 13 & 0 & 0 \\
21 & 34 & 0 & 55 \\
\end{bmatrix}
\]
The minimum rank of a digraph $\Gamma$ is defined as
\[ mr(\Gamma) = \min \{ \text{rank}(A) : A \in \mathcal{M}(\Gamma) \} \].
The minimum rank of a digraph $\Gamma$ is defined as

$$mr(\Gamma) = \min\{\text{rank}(A) : A \in \mathcal{M}(\Gamma)\}.$$
The minimum rank of a digraph $\Gamma$ is defined as

$$mr(\Gamma) = \min\{\text{rank}(A) : A \in \mathcal{M}(\Gamma)\}.$$ 

The maximum nullity of a digraph $\Gamma$ is defined as

$$M(\Gamma) = \max\{\text{null}(A) : A \in \mathcal{M}(\Gamma)\}.$$
The minimum rank of a digraph $\Gamma$ is defined as

$$mr(\Gamma) = \min \{ \text{rank}(A) : A \in \mathcal{M}(\Gamma) \}.$$ 

The maximum nullity of a digraph $\Gamma$ is defined as

$$M(\Gamma) = \max \{ \text{null}(A) : A \in \mathcal{M}(\Gamma) \}.$$ 

Theorem

For any digraph $\Gamma$, $M(\Gamma) + mr(\Gamma) = |\Gamma|$. 

Combinatorial Matrix Theory (ISU) October 2013 4 / 20
The minimum rank of a digraph $\Gamma$ is defined as 
$$mr(\Gamma) = \min\{\text{rank}(A) : A \in M(\Gamma)\}.$$

The maximum nullity of a digraph $\Gamma$ is defined as 
$$M(\Gamma) = \max\{\text{null}(A) : A \in M(\Gamma)\}.$$

Theorem

For any digraph $\Gamma$, $M(\Gamma) + mr(\Gamma) = |\Gamma|$.

Theorem

For any digraph $\Gamma$, $mr(\Gamma) \neq n$ and $M(\Gamma) \neq 0$. 
The color change rule for digraphs states that given a blue vertex $b$ and a white vertex $w$, $b$ forces $w$ to turn blue if $w$ is the only white out-neighbor of $b$. 
The color change rule for digraphs states that given a blue vertex $b$ and a white vertex $w$, $b$ forces $w$ to turn blue if $w$ is the only white out-neighbor of $b$.

$b$ cannot force $w$
The color change rule for digraphs states that given a blue vertex $b$ and a white vertex $w$, $b$ forces $w$ to turn blue if $w$ is the only white out-neighbor of $b$.

$b$ cannot force $w$

$b$ can force $w$
The zero forcing number, $Z(\Gamma)$, is the minimum number of vertices that need to be colored blue in order to force the rest of the graph to be colored blue through the color change rule.
The zero forcing number, $Z(\Gamma)$, is the minimum number of vertices that need to be colored blue in order to force the rest of the graph to be colored blue through the color change rule.
The zero forcing number, \( Z(\Gamma) \), is the minimum number of vertices that need to be colored blue in order to force the rest of the graph to be colored blue through the color change rule.
The zero forcing number, $Z(\Gamma)$, is the minimum number of vertices that need to be colored blue in order to force the rest of the graph to be colored blue through the color change rule.

For any digraph $\Gamma$, $M(\Gamma) \leq Z(\Gamma)$ (Barioli et al., 2008)(Hogben, 2010).
Given a zero forcing set and a corresponding chronological list of forces, a **backward arc** is any arc \((u, v) \in E(\Gamma)\) such that \(v\) is forced before \(u\). A **forward arc** is any arc that is not a backward arc.
A path \((v_1, \ldots, v_k)\) in a digraph \(\Gamma\) is **Hessenberg** if it is a path that does not contain any arc of the form \((v_i, v_j)\) with \(j > i + 1\).

**Theorem (Hogben, 2010)**

\[ Z(\Gamma) = 1 \text{ if and only if } \Gamma \text{ is a Hessenberg path.} \]
A path cover of $\Gamma$ is a set of vertex disjoint Hessenberg paths that includes all vertices of $\Gamma$. 
Path Cover

- A **path cover** of $\Gamma$ is a set of vertex disjoint Hessenberg paths that includes all vertices of $\Gamma$.
- The **path cover number**, $P(\Gamma)$, is the minimum number of paths in a path cover for $\Gamma$. 

For any digraph $\Gamma$, $P(\Gamma) \leq Z(\Gamma)$ (Hogben, 2010).
A path cover of $\Gamma$ is a set of vertex disjoint Hessenberg paths that includes all vertices of $\Gamma$.

The path cover number, $P(\Gamma)$, is the minimum number of paths in a path cover for $\Gamma$.

a path cover for $\Gamma$ with $P(\Gamma) = 2$
A path cover of $\Gamma$ is a set of vertex disjoint Hessenberg paths that includes all vertices of $\Gamma$.

The path cover number, $P(\Gamma)$, is the minimum number of paths in a path cover for $\Gamma$.

A path cover for $\Gamma$ with $P(\Gamma) = 2$

For any digraph $\Gamma$, $P(\Gamma) \leq Z(\Gamma)$ (Hogben, 2010).
Digraph of two parallel Hessenberg paths

A Parallel Hessenberg Path

Adding an illegal arc
Important Theorems

Theorem (Berliner et al., Under Review)

\[ Z(\Gamma) = 2 \text{ if and only if } \Gamma \text{ is a digraph of two parallel Hessenberg paths.} \]

Theorem (Hogben, 2010)

Suppose \( \Gamma \) is a digraph and \( \mathcal{F} \) is a chronological list of forces of a zero forcing set \( B \). A maximal forcing chain is a Hessenberg path.
Our Question

What is the maximum number of arcs in a digraph with \( n \) vertices and a given zero forcing number \( k \)?
Maximum Arc Digraph

$|E| = 36$, $|\Gamma| = 7$ and $Z(\Gamma) = 3$
Maximum Arc Digraph

$|E| = 36$, $|\Gamma| = 7$ and $Z(\Gamma) = 3$

$$\sum_{i<j}^{k} n_i n_j + \left( \sum_{i=1}^{k} n_i \right) (k - 1) - \binom{k}{2} + \sum_{i=1}^{k} \left[ \binom{n_i}{2} + (n_i - 1) \right]$$
Given $n_i$ vertices in the $i$-th forcing chain and $Z(\Gamma) = k$:

$$\sum_{i<j}^k n_i n_j$$
Given \( n_i \) vertices in the \( i \)-th forcing chain and \( Z(\Gamma) = k \): 

\[ + \left( \sum_{i=1}^{k} n_i \right) (k - 1) \]
Given $n_i$ vertices in the $i$-th forcing chain and $Z(\Gamma) = k$:

$\binom{k}{2}$
Given $n_i$ vertices in the $i$-th forcing chain and $Z(\Gamma) = k$:

$$+ \sum_{i=1}^{k} \left[ \binom{n_i}{2} + (n_i - 1) \right]$$
Formulation

Given $n_i$ vertices in the $i$-th forcing chain and $Z(\Gamma) = k$:

$$\sum_{i<j}^k n_i n_j + \left( \sum_{i=1}^k n_i \right) (k - 1) - \binom{k}{2} + \sum_{i=1}^k \left[ \binom{n_i}{2} + (n_i - 1) \right]$$
Theorem

For a digraph $\Gamma$ of order $n$ with $Z(\Gamma) = k$, 

$$|E| \leq \binom{n}{2} - \binom{k}{2} + k(n - 1)$$
Independence of Distribution of Vertices

Given $|\Gamma| = n$ and $Z(\Gamma) = k$, the maximum number of arcs is independent of the distribution of the vertices into each of the $k$ forcing chains.

$|\Gamma| = 7$ and $Z(\Gamma) = 3$
$|E(\Gamma)| = 36$
Maximum Nullity of a Maximum Arc Digraph

Theorem

If $\Gamma$ is a digraph with the maximum number of arcs (by our construction), then $M(\Gamma) = Z(\Gamma)$.

$\Gamma$ realizing the maximum number of arcs

Here $Z(\Gamma) = 2$ and $M(\Gamma) = 2$. 

Family of matrices corresponding to $\Gamma$
References


Acknowledgments

Thank you to:

- The National Science Foundation (NSF DMS 0750986)
- Iowa State University
- Leslie Hogben, Adam Berliner, Travis Peters, Michael Young, and Nathan Warnberg
- Joshua Carlson, Jason Hu, Katrina Jacobs, Kathryn Manternack