An Open Problem in the Combinatorics of Macdonald Polynomials

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Abstract
Although a combinatorial interpretation of Macdonald polynomials involving fillings of Ferrers diagrams of partitions has been known since 2004, a general combinatorial proof of a well-known symmetry property of these polynomials remains elusive. Our paper frames the problem and discusses an already-solved special case, that of one-dimensional (single-row or single-column) Ferrers diagrams, and then give our solution for a new special case, that of hook-shaped Ferrers diagrams with standardized (non-multiset) fillings. We then discuss remaining issues.

1 Introduction

1.1 Preliminary Ideas and Definitions
Before we can state our open problem in a meaningful way, we need to establish some properties of the combinatorial objects that we use to define and study the Macdonald polynomials. The following definitions review some of the basic combinatorial concepts that will become important in our discussion. We also give some examples of how these definitions may be applied.

Definition 1.1 A partition of a positive integer \( n \) is a nonincreasing sequence \( \mu = (\mu_1, \mu_2, ...) \) of positive integers whose sum is \( n \). For instance, for \( n = 11 \), possible partitions include \((4, 3, 2, 2)\); \((6, 5)\); and \((1, 1, 1, 1, 1, 1, 1, 1, 1, 1)\). A Ferrers diagram of a partition \( \mu \) will be defined as several stacked rows of left-justified boxes with the number of boxes in row \( i \) (counting the bottom row as row 1) given by \( \mu_i \).

Note that we are using what is sometimes called the French convention for Ferrers diagrams (the number of boxes in the bottom row, rather than in the top row, corresponds to the first term of the partition).

Example 1.2 The Ferrers diagram of \( \mu = (4, 3, 2, 2) \) is

\[
\begin{array}{c|c|c|c|c}
\hline
| & | & & |
|---|---|---|---|
| & & & |
| & & & |
| & & & |
| & & & |
\hline
\end{array}
\]
Definition 1.3 The conjugate of a partition \( \mu \), denoted by \( \mu' \), is the partition whose \( i \)th column from the left contains \( \mu_i \) boxes.

More intuitively, this can be thought of as "flipping" a partition's Ferrers diagram over the line \( y = x \). The Ferrers diagram of \( \mu \) and \( \mu' \) are reflections of each other in this sense, as the example below shows.

**Example 1.4** The conjugate of \( \mu = (4,3,2,2) \) is \( \mu' = (4,4,2,1) \), as seen in their Ferrers diagrams below.

![Ferrers Diagrams](image)

\( \mu = (4,3,2,2) \)

\( \mu' = (4,4,2,1) \)

We will want to be able to use Ferrers diagrams to represent polynomials. The following two definitions give us a means to do this.

**Definition 1.5** A filling \( \sigma \) of a Ferrers diagram \( \mu \) is the assignment of a positive integer to each box in the diagram. We use the term content to refer to the multiset of integers appearing in a filling.

**Example 1.6** Some possible fillings of the diagram of \( (4,3,2,2) \) with different content are

\[
\begin{array}{cccc}
1 & 1 & 0 & \\
9 & 8 & & \\
7 & 6 & 5 & \\
4 & 3 & 2 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
6 & 4 & & \\
2 & 4 & & \\
3 & 3 & 2 & \\
1 & 4 & 1 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
2 & 2 & & \\
2 & 2 & & \\
2 & 2 & 2 & \\
2 & 2 & 2 & 2 \\
\end{array}
\]

Using the following correspondence, the content of a filling of a Ferrers diagram represents a monomial in several variables.

**Definition 1.7** Let \( \sigma \) be a filling of a Ferrers diagram \( \mu \). Then \( x^\sigma \) indicates the monomial

\[
\prod_{i=1}^{\infty} x_i^{k_i}
\]

where \( k_i \) is the multiplicity of the integer \( i \) in the content of \( \sigma \).

Note that the monomial so defined depends only on the content of \( \sigma \), and not on any aspect of the shape of \( \mu \) apart from the total number of boxes.

**Example 1.8** In the case of the filling \( 
\begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{array}
\)

we have the monomial

\[
x^\sigma = x_1^2 x_2^4 x_3 x_5.
\]

One final definition is necessary before we can introduce our problem.
**Definition 1.9** A **statistic** is a combinatorial rule that associates each element of some set of combinatorial objects with a nonnegative integer.

When considering the sets of fillings of a Ferrers diagram, some simple statistics might be the number of distinct integers appearing in the filling, or the greatest multiplicity of a single integer in the filling.

The two slightly more involved statistics that we will be considering in this paper are called the **major index** and the **inversion number**. They are defined on a filling $\sigma$ of a Ferrers diagram for a partition $\mu$ and are denoted $\text{maj}(\sigma, \mu)$ and $\text{inv}(\sigma, \mu)$ respectively. We will first define these statistics on permutations (which, we will show, can be thought of as fillings of one-dimensional Ferrers diagrams), and later we will extend this definition to Ferrers diagrams of arbitrary shape.

### 1.2 Macdonald Polynomials

Macdonald gave an algebraic definition of the Macdonald polynomials in 1987. More recently, in 2004, Haglund, Haiman, and Loehr proved the equivalence of a combinatorial interpretation that had been earlier conjectured by Haglund [3]. We will take this combinatorial interpretation as our definition of the Macdonald polynomials.

**Definition 1.10** Let $\mu$ be an arbitrary partition. The **Macdonald polynomial** $C_\mu$ is given by

$$C_\mu(x_1, x_2, \ldots; q, t) = \sum_{\sigma} x^{\sigma} t^{\text{maj}(\sigma, \mu)} q^{\text{inv}(\sigma, \mu)}$$

where the sum is over all possible fillings $\sigma$ of $\mu$.

Macdonald polynomials have long been known to be symmetric in $x_1, x_2, \ldots$, and they are also known to possess the following additional symmetry property.

$$C_\mu(x_1, x_2, \ldots; q, t) = C_\mu'(x_1, x_2, \ldots; t, q)$$

Note that one clear consequence of this symmetry in light of (1) is that the distribution of major indices on fillings of a particular Ferrers diagram with a particular content must be the same as the distribution of inversion numbers on fillings of the conjugate diagram with that same content, and vice versa. However, the search for a combinatorial interpretation of this symmetry relation remains an open problem, with only a few special cases solved [2].

In this paper we will first review Foata and Schützenberger’s work [1] on major index and inversion number of permutations. After showing that this represents a special case of our problem, we generalize the problem and presenting our own solution for another special case. We conclude by discussing the areas for future work on the problem, including the limitations of our approach for seeking more general results.

### 2 Special Case: Statistics on Permutations

We now step away from the broad world of Ferrers diagrams and focus on the special case of permutations. We begin by establishing the definitions of inversion number and major index on permutations of multisets. Then, we review Foata and Schützenberger’s bijection $\phi$ on permutations and show how it maps major index to inversion number [1].
2.1 Permutations

Definition 2.1 A permutation of a set or multiset \( A = \{a_1, a_2, \ldots, a_n\} \) is an ordering of the elements of \( A \).

It is common to use the term "permutation" to refer to a permutation of \([n]\), the positive integers from 1 to \( n \). Here we define permutation more broadly to apply to sets and multisets. In this paper we will focus on permutations of multisets of positive integers. We can associate a permutation of such a multiset of size \( n \) with a permutation of \([n]\) through standardization.

Definition 2.2 Consider the multiset \( B = \{b_1, \ldots, b_1, b_2, \ldots, b_2, \ldots, b_k, \ldots, b_k\} \), where \( b_1 < b_2 < \ldots < b_k \) are positive integers and \( c_i \) is the multiplicity of \( b_i \) for each \( i \). Let \( \sigma \) be a permutation of \( B \). Then the standardization of \( \sigma \), \( \text{std}(\sigma) \), is a permutation of \(|B|\) constructed by replacing the copies of \( b_i \) with \( \sum_{j=1}^{i-1} c_j + 1, \sum_{j=1}^{i-1} c_j + 2, \ldots, \sum_{j=1}^{i-1} c_j + c_i \) from left to right for each \( i \).

While the definition of standardization is rather technical, the intuition behind the process is fairly straightforward. We wish to convert a permutation of a multiset into a permutation with distinct entries in a way that preserves the essence of the original permutation. If one entry is greater than another entry, they should maintain this relationship in the standardized permutation. Our method of standardization enforces this by assigning the copies of \( b_i \) the first \( c_i \) integers (starting at 1), the copies of \( b_2 \) the next \( c_2 \) integers, the copies of \( b_3 \) the next \( c_3 \) integers, etc. Thus, if \( b_i < b_j \), then each copy of \( b_i \) is assigned a value smaller than any value assigned to a copy of \( b_j \).

If two entries are equal, it is most natural to assign a smaller value to the one we encounter first (farther to the left). Again, our method of standardization enforces this by assigning equal entries increasing values from left to right.

Example 2.3 The standardization of \( \sigma = 42472 \) is \( 31452 \).

There are many important statistics that can be calculated on permutations. In this paper, we consider the two statistics that appear in the combinatorial definition of Macdonald polynomials: the inversion number and major index.

Definition 2.4 An inversion pair of a permutation \( \sigma \) is a pair of indices \( i < j \) such that \( \sigma_i > \sigma_j \).

Intuitively, an inversion occurs when two entries are out of order in a permutation. Note that standardization does not affect whether or not a pair of indices is an inversion pair. To see this, first note that if \( \{i, j\} \) is an inversion pair in \( \sigma \) then \( \sigma_i > \sigma_j \). If two entries are already distinct, standardization does not change their relative order, so \( \text{std}(\sigma_i) > \text{std}(\sigma_j) \) and \( \{i, j\} \) is an inversion pair in \( \text{std}(\sigma) \). If \( \{i, j\} \) is not an inversion pair in \( \sigma \) then \( \sigma_i \leq \sigma_j \). Again, if the two entries are already distinct, standardization does not change their relative order. If the two entries are equal, \( \sigma_i \) will be assigned a smaller value than \( \sigma_j \) because it appears farther to the left. Either way, \( \text{std}(\sigma_i) < \text{std}(\sigma_j) \) and \( \{i, j\} \) is not an inversion pair in \( \text{std}(\sigma) \).

Definition 2.5 The inversion number (\( \text{inv} \)) of a permutation \( \sigma \) is the number of inversion pairs in \( \sigma \).

Example 2.6 The inversion number of \( \sigma = 362514 \) is 9 because \( \sigma \) has 9 inversion pairs: \( \{1, 3\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 6\}, \{3, 5\}, \{4, 5\}, \{4, 6\} \).

Definition 2.7 A descent of a permutation \( \sigma \) is an index \( i \) such that \( \sigma_i > \sigma_{i+1} \).
Intuitively, a descent occurs when an entry is larger than the entry immediately after it. Note that $i$ is a descent exactly when $\{i, i + 1\}$ is an inversion pair. This implies that standardization does not affect whether an index $i$ is a descent.

**Definition 2.8** The major index $(\text{maj})$ of a permutation $\sigma$ is the sum of the descents of $\sigma$.

**Example 2.9** The major index of $\sigma = 3, 6, 2, 5, 1, 4$ is 6 because $\sigma$ has 2 and 4 as descents and $2 + 4 = 6$.

It is important to note that standardizing a permutation affects neither the inversion number nor the major index. As argued above, standardization does not affect whether or not a pair is an inversion pair, so it does not affect the number of inversion pairs - the inversion number. It also does not affect whether or not an index is a descent, so it does not affect the sum of the descents - the major index. The preservation of both statistics is further evidence that our method of standardization is well-chosen.

### 2.2 Foata’s $\phi$ Function on Permutations

In [1], Foata and Schützenberger introduce an invertible function $\phi$ on permutations that takes major index to inversion number. That is, for a permutation $\sigma$, $\text{maj}(\sigma) = \text{inv}(\phi(\sigma))$. Note that $\phi$ does not take inversion number to major index. To address this, Foata and Schützenberger present another function $\psi = i \phi (i \phi)^{-1}$ on permutations of $[n]$, where $i$ is the group-theoretic inverse of a permutation. $\psi$ is an involution that reverses major index and inversion number, but it cannot be extended to permutations of multisets, so we will end our discussion of it here.

**Definition 2.10** Let $\sigma$ be a permutation. $\phi(\sigma)$ is calculated by running the following algorithm.

```
image = $\sigma_1$
For $i$ from 2 to length($\sigma$):
If last entry of image > $\sigma_i$:
  Draw a line after each entry of image that is > $\sigma_i$;
Else if last entry of image ≤ $\sigma_i$:
  Draw a line after each entry of image that is ≤ $\sigma_i$;
In each compartment move the last entry to the beginning
image = image with $\sigma_i$ appended to end
Return image
```

**Example 2.11** Let $\sigma = 362514$. The following is a run of the algorithm for $\phi$ to compute $\phi(\sigma)$.

```
3 | 6
3|6 | 2
3|6 2| 5
3|2|6|5 | 1
3|2|6|5|1 | 4
3 2 1 6 5 4
```

The algorithm gives $\phi(\sigma) = 321654$. Recall that $\text{inv}(\sigma) = 9$ and $\text{maj}(\sigma) = 6$, and note that $\text{inv}(\phi(\sigma)) = 6$ and $\text{maj}(\phi(\sigma)) = 12$. We see that $\phi$ sends $\text{maj}$ to $\text{inv}$ but does not send $\text{inv}$ to $\text{maj}$.

**Definition 2.12** Let $\sigma$ be a permutation. $\phi^{-1}(\sigma)$ is calculated by running the following algorithm.
For $i$ from 1 to length($\sigma$) - 1

$s = \text{last entry of input}$

$input = \text{input with } s \text{ removed}$

$image = \text{image with } s \text{ appended to beginning}$

If first entry of input > $s$:

Draw a line before each entry of input that is > $s$

Else if first entry of image ≤ $s$:

Draw a line before each entry of input that is ≤ $s$

In each compartment move the first entry to the end

$image = \text{image with only remaining entry of input appended to beginning}$

Return image

Example 2.13 This time, let $\sigma = 321654$. The following is a run of the algorithm for $\phi^{-1}$.

| 3 2 1 6 5 4 |
| 3 2 6 | 5 |
| 3 6 | 2 |
| 3 | 6 |
| 3 6 2 5 1 4 |

To see why the algorithm for $\phi^{-1}$ reverses the steps of $\phi$, consider a single iteration of each. Suppose that in the algorithm for $\phi$, the last entry of the current image is greater than the next entry, $\sigma_i$. (The argument is similar in the case that the last entry of the current image is less than or equal to the next entry.) We draw lines after each entry of the image that is greater than $\sigma_i$. Each compartment will have some (0 or more) entries less than or equal to $\sigma_i$ followed by a single entry greater than $\sigma_i$. We move the last entry in each compartment to the beginning and append $\sigma_i$ to the end of the current image. If we then run an iteration of $\phi^{-1}$, we would first remove the last entry ($\sigma_i$) and call it $s$. The first entry of the input will be greater than $s$ because in the iteration of $\phi$ in each compartment, the entry greater than $s$ was moved from the end to the beginning. When we draw lines before each entry greater than $s$, each compartment will contain one entry greater than $s$ followed by some (0 or more) entries less than or equal to $s$. These compartments will be the same compartments as in the iteration of $\phi$, and moving the first entry to the end will reestablish the original order. Since this holds for an individual pair of iterations, running the algorithm for $\phi^{-1}$ will undo each step of the algorithm for $\phi$, confirming that they are inverses.

We now show that $\phi$ sends major index to inversion number. Suppose we are running the algorithm and are about to add $\sigma_{i+1}$ to the current image. The last entry of the current image at this point is $\sigma_i$, because we just appended it to the end on the previous iteration. First consider the case in which $\sigma_i > \sigma_{i+1}$. This means that $i$ is a descent in $\sigma$ and contributed $i$ to the major index. We will show that adding $\sigma_{i+1}$ to the image adds $i$ to the inversion number. Suppose there are $k$ entries in the current image that are greater than $\sigma_{i+1}$. Each compartment will end with one of these $k$ entries. Appending $\sigma_{i+1}$ to the end of the current image will create $k$ inversion pairs, one for each compartment. There are $i - k$ entries less than or equal to $\sigma_{i+1}$ and thus not at the end of any compartment. Moving the last element ($> \sigma_{i+1}$) in a compartment to the beginning, creates an inversion pair with each of the other elements ($\leq \sigma_{i+1}$) in the compartment. In total, moving the elements in each compartment creates $i - k$ new inversion pairs. Putting these together, running this iteration of the algorithm for $\phi$ will create $k + i - k = i$ new inversion pairs. The process contributes $i$ to the inversion number of the image.
Example 2.14 To see how this works, consider the example in which the current image is 49217 and the next entry is 6. This means that the index 5 was a descent in the original permutation, contributing 5 to the major index. We draw lines to create compartments: 4 9[2 1 7]. Note that 9 > 6 and 7 > 6, so appending 6 will create 2 inversion pairs. We then move the last entry in each compartment: 9 4[7 2 1]. Moving 9 creates an inversion pair with the 4 and moving 7 creates two inversion pairs, one with the 2 and one with the 1. As desired, there are 5 new inversion pairs, contributing 5 to the inversion number.

Now consider the case in which $\sigma_i \leq \sigma_{i+1}$. This means that $i$ is not a descent in $\sigma$ and contributed 0 to the major index. We will show that adding $\sigma_{i+1}$ to the image adds 0 to the inversion number. Suppose that there are $k$ entries in the current image that are less than or equal to $\sigma_{i+1}$. Each compartment will end with one of these $k$ entries. There are $i-k$ entries greater than $\sigma_{i+1}$, and appending $\sigma_{i+k}$ will create $i-k$ new inversion pairs. The last entry in a compartment is less than each of the other entries in its compartment, so moving it to the beginning will undo an inversion pair with each of these other entries. Then, in total moving the elements in each compartment gets rid of $i-k$ old inversion pairs. Putting these together, running this iteration will create $i-k$ new inversion pairs and get rid of $i-k$ old ones for a net total of 0. The process contributes 0 to the inversion number of the image.

Example 2.15 To see how this works, consider the example in which the current image is 32651 and the next entry is 4. This exact situation occurs in the last step of the above example of the algorithm for $\phi$. Since $1 \leq 4$, the index 5 was not a descent in the original permutation, and there is no contribution to the major index. We draw lines to create compartments: 3[2]6 5 1. Note that 6 > 4 and 5 > 4, so appending 4 will create 2 inversion pairs. We then move the last entry in each compartment: 3[2]1 6 5]. The 1 was in inversion pairs with the 5 and 6, but moving it gets rid of these inversion pairs. As desired, there are 2 new inversion pairs and 2 old inversion pairs were undone, for a net total of 0 new inversion pairs and no contribution to the inversion number.

We mentioned before that, unlike $\psi$, $\phi$ can be extended to multiset permutations. We have already laid out the groundwork for this by specifying what to do in the algorithms for $\phi$ and $\phi^{-1}$ in the case of equal elements. Here we argue that running $\phi$ on a multiset permutation is equivalent to running $\phi$ on its standardization and then “unstandardizing” back to the original multiset. It is important to note that “unstandardization” is only defined when entries that will be assigned the same value are in increasing order from left to right. Otherwise, the process might assign entries involved in an inversion pair or descent the same value, thus deleting that inversion pair or descent. When we standardize, the major index (and inversion number) does not change and the equal entries are assigned values in increasing order from left to right. As the algorithm for $\phi$ runs, it changes the order of two entries exactly when the value of the new entry is between their values. Since “equal” entries are arranged in increasing order, $\phi$ never changes that order of two “equal” entries. Thus, all of the entries that will be assigned equal values are in the correct order after running $\phi$ and we can unstandardized back to the original multiset. Since standardization and unstandardization (when defined) do not affect major index and inversion number, $\text{maj}(\sigma) = \text{maj}(\text{std}(\sigma)) = \text{inv}(\phi(\text{std}(\sigma))) = \text{inv}(\phi(\sigma))$, as desired.

3 Statistics on Ferrers Diagrams

The time is now ripe to return to our original question. Before we do this, however, we must define the major index and inversion number on Ferrers diagrams, and show how Foata’s map on permutations represents a special case of our problem.
3.1 Major Index

**Definition 3.1** Given a filling $\sigma$ of a Ferrers diagram of a partition $\mu$, a descent is a box in the Ferrers diagram whose integer entry is strictly greater than the entry of the box immediately below it. Let the set of all descents in $\sigma$ be denoted by $\text{Des}(\sigma)$. For a block $s$, its index, denoted $i_s$, is the number of blocks above $s$ in its own column, plus one (alternatively, and perhaps more intuitively the index may be thought of as the row of the block, counting from the top, with the top row having index 1).

Define

$$\text{maj}(\sigma, \mu) = \sum_{s \in \text{Des}(\sigma)} i_s,$$

the sum of the indices of the descents of $\sigma$.

**Example 3.2** The descents of the filling of the $(4, 3, 2, 2)$ below are indicated in boldface. We use the indices of these descents to calculate the major index of the filling.

```
6 4 4
2 4
3 3 2
1 4 1 1
```

The index of the descent containing 6 is 1, the index of the descent containing 4 is 2, the index of the descent containing 3 is 3, and the index of the descent containing 2 is 1. Therefore, $\text{maj}(\sigma, \mu) = 7$.

3.2 Inversion Number

Haglund gives two equivalent definitions of inversion number in his work on the combinatorics of Macdonald polynomials [2]. We will present one of them here.

**Definition 3.3** Three boxes $u$, $v$, and $w$ in a filling of a Ferrers diagram are defined to be an inversion triple if they satisfy the following rules:

- $v$ is immediately below $u$ ($v$ is in the same column as $u$ and exactly one row below $u$)
- $w$ must be in the same row as $u$ and to the right of $u$ (though not necessarily immediately to the right)
- The path described by moving through all three boxes starting with the one with the smallest entry and finishing at the one with the largest entry is counterclockwise. (Break ties by considering equivalent entries in higher rows to always be smaller than in lower rows, and within a row consider equivalent entries to the left smaller than entries to the right.)

Note: for purposes of calculating inversion number, we suppose the existence of a “basement” row below the lowest row whose boxes are filled with virtual infinity symbols. Thus, an inversion triple can exist including only two boxes in the bottom non-basement row if the left box in the pair has a greater entry than the right box.

We now say that $\text{inv}(\sigma, \mu)$ is equal to the number of inversion triples in $\sigma$.

**Example 3.4** In the following filling, there are six inversion triples: $[(1, 4), (2, 4), (1, 3)], [(1, 3), (2, 3), (1, 2)], [(1, 2), (2, 1), (3, 1), (4, 1)]$; therefore the $\text{inv}(\sigma, \mu) = 6$.

```
6 4 4
3 4
3 3 2
1 4 1 1
```
3.3 Standardization of Fillings Ferrers Diagrams

Multiset fillings of Ferrers diagrams can be standardized to set fillings in much the same way that permutations can, with some tweaking.

**Definition 3.5** The reading word of a filling $\sigma$ of the Ferrers diagram of a partition $\mu$ is the multiset permutation consisting of the entries in the filling read from left to right, from the top row to the bottom row (as you would read a book). The standardization of $\sigma$ is the standardization of its reading word.

The following example shows how a filling of a Ferrers diagram may be standardized.

**Example 3.6** The standardization of

\[
\begin{array}{ccc}
6 & 4 & \\
2 & 4 & \\
3 & 3 & 2 \\
1 & 4 & 1 & 1 \\
\end{array}
\]

is

\[
\begin{array}{ccc}
1 & 1 & 8 \\
4 & 9 & \\
6 & 7 & 5 \\
1 & 2 & 10 & 3 \\
\end{array}
\]

Note that standardizing in this way will never affect the major index or inversion number; descents and inversion triples are preserved and no new descents or inversion triples can be created.

3.4 Permutations as Fillings of Special Ferrers Diagrams

Note that a permutation or multiset permutation can be viewed as a filling of either a vertical or a horizontal Ferrers diagram. For instance, the permutation $\pi = (4, 3, 1, 5, 2)$ can be represented as either

\[
\begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}
\]

or

\[
\begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}
\]

We claim that with this interpretation in mind, $\phi$ actually represents a solution to a special case of our problem.

**Theorem 3.7** $\phi$ (with $\phi^{-1}$) is a bijection that takes fillings of one-dimensional Ferrers diagrams to fillings of their conjugate diagrams with the same content and that reverses major index and inversion number.

**Proof** In a column permutation, $\text{inv}(\sigma, \mu)$ will be 0 since there cannot be any inversion triples, and $\text{maj}(\sigma, \mu) = \text{maj}(\pi)$ (this follows easily from the second interpretation of the index of a box in a Ferrers diagram provided above). In a row permutation, because there cannot be descents we have $\text{maj}(\sigma, \mu) = 0$; meanwhile we have $\text{inv}(\sigma, \mu) = \text{inv}(\pi)$, a trivial consequence of the preceding characterization of inversion triples in the bottom row.

Because these ”row” and ”column” Ferrers diagrams are conjugate shapes of one another, it is well-defined to use $\phi$ to send a column to its conjugate row and $\phi^{-1}$ to send a row to its conjugate column. If $\phi$ is used to send a column permutation to a row permutation, then the major index of the column will become the inversion number of the row by the property of $\phi$, while the zero inversion number of the column will become the zero major index of the row. The reverse of all this is true if $\phi^{-1}$ is used to send a row permutation to a column. Therefore $\phi$ is an explicit bijection from fillings of a partition (given that the partition is of the form $\mu = (n)$ or of the form $\mu = (1, 1, 1, 1...)$) to fillings of its conjugate partition that reverses the statistics of interest. \qed
The Macdonald polynomials corresponding to this class of Ferrers diagram will have the form

\[ C_\mu(x_1, x_2, \ldots; q, t) = \sum_\sigma x^{\sigma \ell_{maj}(\sigma, \mu)} \]

in the case of columns and

\[ C_\mu(x_1, x_2, \ldots; q, t) = \sum_\sigma x^{\sigma \ell_{inv}(\sigma, \mu)} \]

in the case of rows. The missing \( q \) and \( t \) terms respectively represent the zero value of the inversion number and major index on columns and rows respectively. Our goal now is to attempt to make use of this result to find similar bijections on a broader class of Ferrers diagram shapes that will combinatorially establish the symmetry for further Macdonald polynomials.

\section{An Involution on Hooks}

We were able to partially develop the desired bijection for shapes known as hooks.

\subsection{Definitions}

We will begin by defining a hook and several important terms relating to it.

\textbf{Definition 4.1} A Ferrers diagram \( S \) is a \textbf{hook} if every row except the bottom has only one box in it. In other words, it is a column with the end joined to the beginning of a row. When filled, \( S \)’s \textbf{corner}, \( \text{cor}(S) \), is the entry which is in both the row and column. The \textbf{column content} of \( S \), which we shall call \( K_S \), is the set of terms which are in the column; similarly, the \textbf{row content} of \( S \), which we shall call \( R_S \), is the set of terms in the row.

We shall also make use of the standardized permutations of the row and column.

\textbf{Definition 4.2} The \textbf{standardization of the column permutation} of \( S \), which we shall call \( \kappa_S \), is the standardization of the reading word of the column taken independently from the row; similarly, the \textbf{standardization of the row permutation} of \( S \), which we shall call \( \rho_S \), is the standardization of the reading word of the row taken independently from the column.

\textbf{Example 4.3} Let

\[
S = \begin{array}{c}
1 \\
4 \\
7 \\
2 \ 5 \ 6 \ 3 \ 8 \\
\end{array}
\]

\( S \) is a hook filled with the terms 1 through 8.

\[ \text{cor}(S) = 2 \]

\[ K_S = \{1, 2, 4, 7\}, R_S = \{2, 3, 5, 6, 8\} \]

and

\[ \kappa_S = 1342, \rho_S = 13425 \]

For the purposes of this section, we will only deal with hooks that are filled with no repeated terms. Without loss of generality, we will treat the terms filing a hook with \( n \) boxes as the integers 1 through \( n \).
4.2 An Equivalence Relation on Hooks

The standardization of the row and column permutations yields a useful equivalence relationship on hooks.

**Definition 4.4** We say that for two hooks $S$ and $V$ of the same shape, $S \approx V$ whenever $\kappa_S = \kappa_V$ and $\rho_S = \rho_V$.

**Example 4.5** If

\[
S = \begin{array}{cccc}
1 & 4 & 7 \\
5 & 6 & 3 & 8
\end{array}
\]

and

\[
V = \begin{array}{cccc}
1 & 5 & 8 \\
2 & 4 & 6 & 3 & 7
\end{array}
\]

$S \approx V$, since

\[\kappa_S = 1342 = \kappa_V\]

and

\[\rho_S = 13425 = \rho_V\]

Within the equivalence classes of $\approx$, the two statistics important to Macdonald polynomials remain constant.

**Theorem 4.6** If $S$ and $V$ are hooks and $S \approx V$ then $\maj(S) = \maj(V)$ and $\inv(S) = \inv(V)$.

To show this, we first need a lemma.

**Lemma 4.7** Let $S$ be a hook. $\maj(S) = \maj(\kappa_S)$ and $\inv(S) = \inv(\rho_S)$

**Proof** As discussed previously, the inversion number of a column is zero, as is the major index of a row. This holds with hooks. Since there can be no descent unless there is at least a stack of two boxes, there are no descents in the row of a hook. Similarly, there has to be a box to the right of a box for there to be an inversion triple, so there can be no inversions in the column of a hook. Thus, the major index of a hook is the same as the major index of its column, and the hook’s inversion number is the same as the inversion number of its row. Furthermore, since standardization does not affect statistics, the row and the column can be independently standardized, without either statistic changing. Finally, as shown earlier, the major index of a column is that of its reading word, and the inversion statistic of a row is also the inversion statistic of its reading word. Altogether, this implies that for a hook $S$, $\maj(S) = \maj(\kappa_S)$ and $\inv(S) = \inv(\rho_S)$.

With this, the proof our theorem is easy.

**Proof** If for two hooks $S$ and $V$, $S \approx V$, then $\kappa_S = \kappa_V$ and $\rho_S = \rho_V$. Accordingly,

\[\maj(S) = \maj(\kappa_S) = \maj(\kappa_V) = \maj(V)\]

and

\[\inv(S) = \inv(\rho_S) = \inv(\rho_V) = \inv(V)\].

\[\square\]
Furthermore, every member of an equivalence class of \( \approx \) will have the same corner entry.

**Theorem 4.8** If \( S \approx V \), then \( \text{cor}(S) = \text{cor}(V) \).

**Proof** Let \( S \approx V \). Thus, \( \kappa_S = \kappa_V \) and \( \rho_S = \rho_V \). Let us say that \( \kappa_S \) is of length \( k \) and \( \rho_S \) is of length \( r \). Accordingly, both \( S \) and \( V \) are \( k + r - 1 \) terms long (since both permutations include the corner). Further, let the last term in \( \kappa_S \) be \( a \) and the first term in \( \rho_S \) be \( b \). These are the terms corresponding to \( \text{cor}(S) \) and \( \text{cor}(V) \). Thus, there are \( a - 1 \) terms smaller and \( k - a \) bigger than \( \text{cor}(S) \) and \( \text{cor}(V) \) in \( \kappa_S \) and similarly \( b - 1 \) terms smaller and \( r - b \) terms bigger than \( \text{cor}(S) \) and \( \text{cor}(V) \) in \( \rho_S \). Therefore, altogether in \( S \) and \( V \), there must be \( a + b - 2 \) terms smaller and \( k + r - a - b \) terms bigger than \( \text{cor}(S) \) and \( \text{cor}(V) \). The only term which this could possibly be true for is \( a + b - 1 \); it has \( a + b - 2 \) terms smaller and \( k + r - a - b \) terms bigger than it in \( S \) and \( V \). Accordingly, \( \text{cor}(S) = \text{cor}(V) = a + b - 1 \).

Using this piece of information, we can figure out the size of a hook’s equivalence class.

**Theorem 4.9** Let \( S \) be a hook with \( \kappa_S \) of length \( k \) and \( \rho_S \) of length \( r \). Furthermore, let \( a \) be the entry in \( \kappa_S \) corresponding to the corner term and \( b \) be the entry in \( \rho_S \) also corresponding to the corner term. Then

\[
\| [S]_{\approx} \| = \binom{a + b - 2}{a - 1} \binom{k + r - a - b}{k - a}
\]

**Proof** For a hook \( V \) to be in \( [S]_{\approx} \), it must have the same standardized row and column permutations. For this to be the case and for the corner entry to remain constant, there have to be the same number of terms larger and smaller than the corner entry in the row and column respectively. Since whatever terms get put in the column determine what is in the row by default, the number of ways to assign smaller and larger terms than the corner to the column need to be counted. There are \( a + b - 2 \) total terms smaller than \( \text{cor}(S) \) in a hook of the same shape, of which \( a - 1 \) must be chosen to go in the column, yielding \( \binom{a + b - 2}{a - 1} \) possibilities. Similarly, there are \( k + r - a - b \) total terms larger than \( \text{cor}(S) \) in a hook of the same shape, of which \( k - a \) must go in the column, yielding \( \binom{k + r - a - b}{k - a} \). Multiplying these two yields all possibilities for assigning larger and smaller terms.

This indicates an important result: for two equivalence classes of hooks of the same shape (or conjugate shape, since the length of \( \kappa \) and \( \rho \) will be switched) to have the same cardinality, they must fulfill one of two conditions. The first is that corner entry of the two classes must be the same. The second is that if we say the corner entry of the first class of hooks is \( a + b - 1 \) as before, and the hook has \( r + k - 1 \) boxes, the corner entry of the second class is \( k + r - a - b + 1 \). This works since there will be \( k + r - a - b \) entries smaller and \( a + b - 2 \) entries bigger than the corner in the hook, yielding \( \binom{a + b - 2}{a - 1} \binom{k + r - a - b}{k - a} \) hooks in both classes. \( k + r - a - b + 1 \) is called the complement of \( a + b - 1 \) in a hook of length \( r + k - 1 \). We shall define this formally shortly.

### 4.3 Reverse and Complement

As our equivalence relationship shows, the statistics of a hook are determined by its equivalence class; this in turn is defined by the standardized row and column permutations. Using this, we will be able to build our involution. However, first we need to define two operations on permutations.

**Definition 4.10** Let \( \pi \) be a permutation. The **reverse** of \( \pi \), \( r(\pi) \), is \( \pi \) read backwards.
Example 4.11 If 
\[ \pi = 13542 \]
then 
\[ r(\pi) = 24531 \]

Definition 4.12 Let \( \pi \) be a permutation of length \( n \) and \( S \) a hook of length \( m \). The **complement** of \( \pi \) is the permutation where each entry \( y \) in \( \pi \) is replaced with \( n - y + 1 \). In essence, this replaces \( y \), which is the \( y \)th smallest entry, with the \( y \)th largest entry. Similarly, if \( x \) is in the hook \( S \), the complement of \( x \), \( c(x) \), is \( m - x + 1 \).

Thus, as mentioned earlier, the complement of \( a + b - 1 \) in a hook of length \( r + k - 1 \) is 
\[ c(a + b - 1) = (r + k - 1) - (a + b - 1) + 1 = r + k - a - b + 1 \]

**Theorem 4.13** Let \( \pi \) be a permutation of length \( n \).
\[ \text{inv}(c(\pi)) = \text{inv}(r(\pi)) = \left( \frac{n}{2} \right) - \text{inv}(\pi) \]

**Proof** There can be either one or no inversion between each pair of terms in a permutation; since there are \( \binom{n}{2} \) total pairs of terms, this means that there are \( \binom{n}{2} \) total possible inversions. If an inversion exists between two terms in \( \pi \), that means that they are out of order, and the larger term precedes the smaller term; if there is not an inversion, the smaller proceeds the larger. In \( r(\pi) \), if two terms were out of order in \( \pi \) they will no longer be and vice versa. Thus, the location of inversions and non-inversions will switch; if there were \( \text{inv}(\pi) \) inversions before, every pair but \( \text{inv}(\pi) \) will be inversions now, yielding the result.

Similarly, since for \( x, y \in \pi, x > y \) implies \( c(x) < c(y) \), there will be an identical switching of inversions and non-inversions, yielding the same result. \( \square \)

This is a valuable property, since it implies that \( \text{inv}(r(c(\pi))) = \text{inv}(\pi) \).

4.4 The Involution

Making use of these two functions on permutations, we will be able to define our involution. Our involution will not only send each \( S \) to a conjugate shaped \( V \) such that the statistics are flipped, but it will map \([S]_\approx\) to \([V]_\approx\). We will do this by defining two functions that send \( \kappa_S \) to \( \kappa_V \) and \( \rho_S \) to \( \rho_V \) such that \( \text{cor}(V) = c(\text{cor}(S)) \).

**Definition 4.14** Let \( S \) be a hook with column and row content \( K_S \) and \( R_S \), and standardized row and column permutations \( \kappa_S \) and \( \rho_S \). Let
\[ Y(\kappa_S) = c(r(\phi(\kappa_S))) \]
and 
\[ Z(\rho_S) = \phi^{-1}(r(c(\rho))). \]

Then \( T(S) \) is the hook with the conjugate shape of \( S \) with column content \( K_{T(S)} = c(R_S) \), row content \( R_{T(S)} = c(K_S) \), column permutation \( \kappa_{T(S)} = Z(\rho_S) \) and row permutation \( \rho_{T(S)} = Y(\kappa_S) \).
Example 4.15 Let
\[
S = \begin{pmatrix}
1 & 4 \\
7 & 2 5 6 3 8
\end{pmatrix}
\]
Thus, the row and column contents are
\[
K_S = \{1, 4, 7, 2\}
\]
\[
R_S = \{2, 5, 6, 3, 8\},
\]
and the row and column permutations are
\[
\kappa_S = 1342
\]
\[
\rho_S = 13425.
\]
We apply \(Z\) to \(\rho_S\) and \(Y\) to \(\kappa_S\) to get the column and row permutations for \(T(S)\):
\[
\kappa_{T(S)} = Z(\rho_S) = 14235
\]
\[
\rho_{T(S)} = Y(\kappa_S) = 3142.
\]
We take the complement of \(R_S\) and \(K_S\) to get \(T(S)\) column and row content:
\[
K_{T(S)} = c(R_S) = \{1, 3, 4, 6, 7\}
\]
\[
R_{T(S)} = c(K_S) = \{2, 5, 7, 8\}
\]
Then, we use the new contents and permutations to fill in \(T(S)\):
\[
T(S) = \begin{pmatrix}
1 & 6 \\
3 & 4 \\
7 2 8 5
\end{pmatrix}
\]
If \(K_{T(S)}\) and \(R_{T(S)}\) do not assign the same entry to the corner however, \(T(S)\) will not be well defined. However, \(Z\) and \(Y\) do not allow for this.

Theorem 4.16 \(T\) is well defined.

Proof Let \(S\) be a hook with column and row content \(K_S\) and \(R_S\), and standardized row and column permutations \(\kappa_S\) and \(\rho_S\). Let \(x = \text{cor}(S)\). Let us say that \(v\) corresponds to \(x\) in \(\kappa_S\) and \(w\) corresponds to \(x\) in \(\rho_S\). Due to the direction columns and rows are read in, \(v\) is the last entry in \(\kappa_S\) and \(w\) is the first entry in \(\rho_S\). Thus, since \(\phi\) and \(\phi^{-1}\) do not change the last term of a permutation, \(v\) would still be the last entry in \(\phi(\kappa_S)\), so \(c(v)\) will be the first entry in \(\rho_{T(S)} = c(\phi(\kappa_S))\). Similarly, the last entry of \(r(c(\rho_S))\) would be \(c(w)\), so it would still be the last entry of \(\kappa_{T(S)} = \phi^{-1}(r(c(\rho_S)))\). Therefore, cor\((T(S))\) would correspond to \(c(v)\) and \(c(w)\).

Since \(x\) was the \(v^{th}\) biggest term in \(K_S\), \(c(x)\) will be the \(c(v)^{th}\) biggest term in \(R_{T(S)} = c(K_S)\). Similarly, \(c(x)\) will be the \(c(w)^{th}\) biggest term in \(K_{T(S)} = c(R_S)\). Thus, both \(\kappa_{T(S)}\) and \(\rho_{T(S)}\) will place \(c(x)\) in the corner, assuring \(T(S)\) is in fact a hook. \(\square\)
Furthermore, beyond being well-defined, $T$ does switch statistics.

**Theorem 4.17** $\text{maj}(S) = \text{inv}(T(S))$ and $\text{inv}(S) = \text{maj}(T(S))$.

**Proof**

$$\text{maj}(S) = \text{maj}(\kappa_S) \quad \text{by the lemma}$$

$$= \text{inv}(\phi(\kappa_S)) \quad \text{as proven for } \phi$$

$$= \text{inv}(c(r(\phi(\kappa_S)))) \quad \text{since } \text{inv}(c(r(\phi(\kappa_S)))) = \text{inv}(\kappa_S)$$

$$= \text{inv}(\rho_{T(S)}) \quad \text{by the definition of } \rho_{T(S)}$$

$$= \text{inv}(T(S)) \quad \text{again by the lemma}$$

and

$$\text{inv}(S) = \text{inv}(\rho_S) \quad \text{by the lemma}$$

$$= \text{inv}(r(c(\rho_S))) \quad \text{since } \text{inv}(c(r(\phi(\kappa_S)))) = \text{inv}(\kappa_S)$$

$$= \text{maj}(\phi^{-1}(r(c(\rho_S)))) \quad \text{as proven for } \phi$$

$$= \text{maj}(\kappa_{T(S)}) \quad \text{by the definition of } \kappa_{T(S)}$$

$$= \text{maj}(T(S)) \quad \text{again by the lemma}$$

$\square$

Finally, we can show that $T$ is an involution, giving us that it is a bijection for free.

**Theorem 4.18** $T$ is an involution.

**Proof** $T$ consists of three parts, all which undo themselves if applied twice. Taking the conjugate is an involution between Ferrers diagrams, so if $T$ is applied twice it will yield the same shape. $Z$ and $Y$ are each others inverses, so if $T$ is applied twice they will undo the changes to the row and column permutations. Finally, taking the complement of contents is an involution, so the applied twice $T$ will undo the changes to the row and column content. Altogether, since $T^2(S)$ has the same shape, row and column permutations, and row and column contents as $S$, it will be the same hook. $\square$

5 Conclusions and Future Work

In the previous section, we only dealt with fillings of hooks with no repeated entries. For such fillings, we established that $T$ is an involution that sends inversion number to major index and major index to inversion number. Our result may be expressed in terms of Macdonald polynomials as follows. Let $\mu$ be a hook and restrict $\sigma$ to permutations of multisets with all distinct entries. Then

$$\sum_{\sigma \text{ with no repeats}} x^{\sigma \text{maj}(\sigma, \mu)} q^{\text{inv}(\sigma, \mu)} = \sum_{\sigma \text{ with no repeats}} x^{\sigma \text{inv}(\sigma, \mu')} q^{\text{maj}(\sigma, \mu')}.$$

Note that in the Macdonald polynomial of the hook $\mu$, the sum is over all possible fillings $\sigma$ of $\mu$. In contrast, in our result the sum is only over those fillings $\sigma$ with no repeated entries.

As demonstrated in the example below, $T$ does not extend to fillings of hooks with repeated entries. $T$ establishes an explicit bijection for hooks with no repeated entries, but its failure on
instances with repeated entries only implies that $T$ is not the explicit bijection we seek on general hooks. An explicit bijection may exist for general hooks exists, and we merely have yet to discover it. A likely candidate would be some modified version of $T$ which would reduce to $T$ in the case of no repeated entries. This is a promising direction for future work. Regardless of the existence of an explicit combinatorial bijection, there certainly is an implicit bijection, since the Macdonald polynomials $C_\mu(x_1, x_2, \ldots; q, t)$ and $C_{\mu'}(x_1, x_2, \ldots; t, q)$ are equal by algebraic proof [3].

**Example 5.1** Consider the following hook with a filling with repeated entries. It has statistics $\text{inv} = 1$ and $\text{maj} = 0$.

```
   1
  2 1
```

To apply $T$ to a filling with repeated entries, we first standardize and then apply $T$ to the standardization. Recall that standardizing does not affect the statistics.

```
   1
  3 2
```

The column content is $K_S = \{1, 3\}$, which standardizes to $\kappa_S = 12$ and the row content is $R_S = \{3, 2\}$, which standardizes to $\rho_S = 21$. Applying $Z$ to $\rho_S$ and $Y$ to $\kappa_S$ gives $\kappa_{T(S)} = Z(\rho_S) = 21$ and $\rho_{T(S)} = Y(\kappa_S) = 12$. The complement contents are $K_{T(S)} = c(R_S) = \{1, 2\}$ and $R_{T(S)} = c(K_S) = \{1, 3\}$. Therefore, the image of the standardization under $T$ is

```
  2
  1 3
```

The image of the standardization has statistics $\text{inv} = 0$ and $\text{maj} = 1$, which is correct. Unfortunately, we cannot “unstandardize” back to the multiset $\{1, 1, 2\}$ by replacing the 1, 2 with 1’s and the 3 with a 2 without modifying the statistics. In particular,

```
  1
  1 2
```

has statistics $\text{inv} = 0$ and $\text{maj} = 0$, which is not the reverse of the statistics of the original filling. This is an example where $T$ fails on a multiset with repeats. Thus, dealing with fillings of hooks with repeats by standardizing, applying $T$, and “unstandardizing” does not the lead to the bijection we seek.

This example shows that $T$ in its present form is limited in its generality as a potential solution to other cases of this problem. Future work in this area might begin with an attempt to plug this hole by finding some variant of $T$ that does not fail in the case of multiset fillings. Further steps might include consideration of other relatively simple classes of Ferrers diagram shapes, especially ones which are their own conjugates or whose conjugates are easily described. As further cases are solved, we will be able to fill in the gaps in our understanding of the combinatorics of Macdonald polynomials.

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