Knotting Probability of Random Polygons
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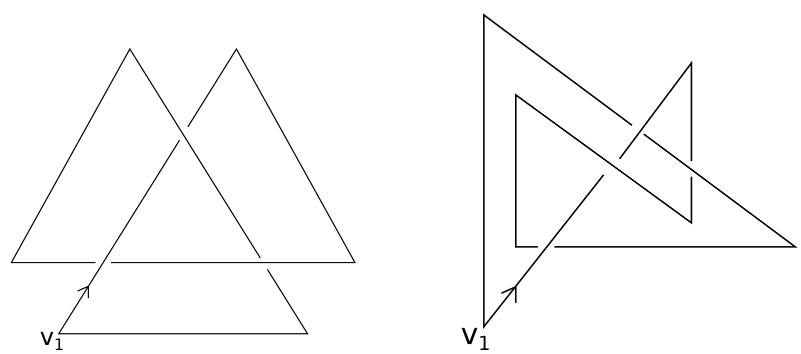
A n-sided polygon, P, in $\mathbb{R}^3$ is a closed, piecewise linear loop with no self-intersections. We label the n vertices $v_1, v_2, \cdots, v_n$ and the n edges $e_1, e_2, \cdots, e_n$ so that e_i connects v_i to v_{i+1} and e_n connects v_n to v_1. Additionally we consider a distinguished vertex, v_1, and a choice of orientation. Now we can view the polygon as a point in $\mathbb{R}^{3n}$ by listing the coordinates of the vertices, starting with v_1 then following the orientation. Due to the condition of no self-intersections, we do not want to consider all points in $\mathbb{R}^{3n}$. We define the discriminant, $\Sigma^{(n)}$, to be all points in $\mathbb{R}^{3n}$ that correspond to non-embedded polygons. Then define the embedding space for rooted, oriented n-sided geometric knots to be $Geo(n) = \mathbb{R}^{3n} - \Sigma^{(n)}$, an open $3n$-dimensional manifold. A path in this space is an isotopy of polygons. So two knots that lie in the same path-component of $Geo(n)$ are defined to be geometrically equivalent. Therefore any knot that is in the same path-component of the regular, planar n-gon is equivalent to the unknot. Adding more structure, we consider polygons of unit edge length. Let $Equ(n)$, a $2n$-dimensional submanifold of $Geo(n)$, be the embedding space for rooted, oriented n-sided equilateral knots.



(A) (B)

Figure 1. Figure (A) shows a 6-sided trefoil knot and figure (B) shows a 7-sided figure-8 knot.

By taking n to be sufficiently large, we can construct geometric knots of any topological knot type. However, if we consider polygons with small numbers of edges we restrict the possible knot types. It is known that you need at least six edges to form a non-trivial knot. Every triangle is planar. A quadrilateral can be folded along the diagonal to become planar. Any pentagon can also be deformed to a planar pentagon. Therefore $Equ(n)$ is connected for $n \leq 5$. The trefoil knot is the only non-trivial knot that can be realized with six edges. The trefoil knot and the figure-8 knot are the only two non-trivial knots that can be formed with seven edges. When $n = 8$, there are seven non-trivial knots that can be realized. Fixing n, what is the probability that a randomly chosen n-sided polygon has knot type K?

In this project, we will investigate the knotting probability for random equilateral hexagons and heptagons. To describe the space of random equilateral polygons, we will look at a structure from symplectic geometry. There exists a measure-preserving set of action-angle coordinates on this space of equilateral polygons, composed of a set of diagonals lengths and angles. We will explore how we can use this coordinate system to prove bounds and generate numerical estimates on the knotting probability.