Quadratics with newly reducible third iterate: how rare is the golden ratio?

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Prerequisites: Math 342 or equivalent experience. You need some experience working with fields and polynomials; you might have gotten this experience by taking certain courses in Budapest, in an REU, or having done some reading on your own. Some familiarity with elliptic curves is helpful but not required.

Description:

The golden ratio (also called the golden mean, golden section, divine proportion, and many other highly laudatory things) is the number

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

It satisfies the equation $\varphi^2 - \varphi - 1 = 0$. It is said that a rectangle whose ratio of length to height is the golden ratio is the most beautiful of all. Human fascination with the golden ratio goes back at least 2400 years, to the Greeks: the sculptor Phidias filled the Parthenon with statues whose proportions reflected the golden ratio. Since that time, it's come up in a bewildering array of scientific and artistic endeavors. This is to say nothing of its mathematical significance; indeed, you've perhaps encountered it as the limiting ratio of consecutive Fibonacci numbers.

Here is yet another outstanding fact about the golden ratio. Consider its associated polynomial $f(x) = x^2 - x - 1$ (recall that φ is a root of this polynomial). It's easy to see that $f(x)$ is irreducible over the rationals, that is, it does not factor as the product of two non-constant polynomials with rational coefficients. The same is true of the polynomial $f(f(x))$, called the second iterate of f. However, the third iterate of f has the unexpected factorization

$$f(f(f(x))) = (x^4 - 3x^3 + 4x - 1)(x^4 - x^3 - 3x^2 + x + 1).$$

This is spectacularly rare. Indeed, among polynomials $x^2 + ax + b$ with a, b integers satisfying $|a| \leq 100,000$ and $|b| \leq 1,000,000,000$, only 8 others have this property.

Open question: are there infinitely many monic quadratic polynomials with integer coefficients whose second iterate is irreducible over the rationals but whose third iterate is not? (Note that irreducibility of the second iterate implies irreducibility of $f(x)$.)

In this project, we will study and attempt to resolve this open question.

One approach to the problem is geometric. The conditions that $f(x) = x^2 + ax + b$ have irreducible second iterate and reducible third iterate force a and b to satisfy a system of polynomial equations in six variables. There are four equations, and so the set of all solutions to this system gives a two-dimensional geometric object, called a surface. Points on this surface with integer coordinates correspond to the a and b we seek. Preliminary calculations suggest that this surface is a parametrized family of elliptic curves – a so-called elliptic surface. It may be possible to find a parametrized family of integer points on this surface (one for each elliptic curve in the family) and that would resolve the open question.

In general, we'd like to learn everything we can about this surface, both in terms of its geometric structure and its arithmetic structure (points with rational and integer coordinates in particular). Some of the work will involve writing computer programs to give simplified equations for the surface and to analyze particular cross-sections of the surface. The work will be mostly algebraic, but with a healthy dose of geometry as well.

In analyzing this surface and thinking about the open problem, we may also come to some insight about whether special properties of the golden ratio are causing the polynomial $x^2 - x - 1$ to exhibit this unusual property.