Bijective Proofs: A Comprehensive Exercise

David Lonoff and Daniel McDonald

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1 In Search of a "Near-Bijection"

Our comps began as a search for a "near-bijection" (a mapping which works on all but a small number of elements) between two sets. The first set, call it $Z(n)$, is the set of solutions to

$$\pm 1 \pm 2 \pm 3 \pm \ldots \pm n = 0.$$

The second set, call it $W(n)$, is the set of solutions to

$$\pm 1 \pm 2 \pm 3 \pm \ldots \pm n = 2.$$

For small values of n, a computer search helped us find the following data:

n	3	4	7	8	11	12	15	16	19
$\lvert Z(n)\rvert$	2	2	8	14	70	124	722	1314	8220
$\lvert W(n)\rvert$	1	2	8	13	69	123	719	1313	8215

Values which are omitted from the table are 0. In general, $|Z(4n+1)| = |Z(4n+2)| = |W(4n+1)| = |W(4n+2)| = 0$ for all $n \geq 0$. This is easy to see by considering parity.

These sequences of numbers seem close enough together that it might seem possible to find a near bijection from one set to the other. However, to see if the pattern persisted, we wanted to compute $|Z(n)|$ and $|W(n)|$ for some larger values of n. We reached a limit to what could be computed in a reasonable amount of time, and so we needed a faster way to generate more numbers in the sequences. Fortunately, the sequences (including the 0 entries) generated by $|Z(n)|$ and $|W(n)|$ are both known (Sequences A063865 and A113036, respectively, in Sloane [10]), which gave us the following data:

n	20	23	24	27	28	31	32	35
$\lvert Z(n)\rvert$	15272	99820	187692	1265204	2399784	16547220	31592878	221653776
$\lvert W(n)\rvert$	15260	99774	187615	1264854	2399207	16544234	31587644	221625505

The fact that the sequences are getting significantly farther apart lead us to give up on our original search for a near bijection between the sets. But we started to notice other patterns. For example, we noticed that $|Z(n)| \geq |W(n)|$, and we attempted (and failed) to prove this. We also noticed that the ratio of $|Z(n)|$ to $|Z(n+1)|$ (when that ratio is defined and not 0) appeared to be slightly above $\frac{1}{2}$. None of the patterns we noticed panned out, and we eventually abandoned this project entirely.

2 In Search of a Bijection

In this section, we discuss a result which cries out for a (constructive) bijective proof. Unfortunately, despite several weeks of attention, we are still unable to find such a proof. Before we state the result, we need some background.

Definition 1 *The* one-line notation *of a permutation* π *of* $[n]$*, where* $[n] = \{1, 2, \ldots, n\}$*, is written* $\pi(1)\pi(2)\ldots\pi(n)$*. We use* S_n *to refer to the set of permutations of* $[n]$ *written in one-line notation.*

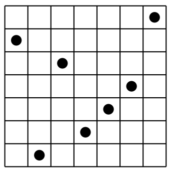
Definition 2 *If* $\pi \in S_n$ *and* $\sigma \in S_k$*, then* π contains σ as a pattern *if some subsequence of* π *of length* k *has the same relative order as* σ*.*

Example 3 *Since* 5287 *has the same relative order as* 2143*, the permutation* 13524867 *contains* 2143*.*

Definition 4 *We say that* π avoids σ *whenever* π *does not contain* σ*.*

Definition 5 *The* diagram *of a permutation* $\pi \in S_n$ *is formed by creating an* $n \times n$ *grid whose rows and columns are labeled from* 1 *to* n *from bottom to top and left to right. A dot is placed in the cell* (i, j) *exactly when* $\pi(i) = j$*.*

Example 6 *The diagram of* $\pi = 6152347$ *is shown below.*



Now we consider two operations on a permutation diagram (and hence on permutations). First, r, the reverse map of a permutation, flips the diagram over a vertical axis (and hence reverses the entries of the permutation). Second, c, the complement map of a permutation, flips the diagram over a horizontal axis (and thus replaces $\pi(i)$ with $n + 1 - \pi(i)$ for each $1 \leq i \leq n$).

We are interested in the permutations whose diagrams are invariant under the rc (reverse-complement) mapping. These are the permutations whose diagrams have 180° rotational symmetry. Denote the set of rotationally symmetric permutations of length n by S_n^{rc}.

Restricted symmetric permutations are permutations in S_n^{rc} which avoid one or more patterns. Let R be a set of patterns. We use $S_n^{rc}(R)$ to denote the set of permutations which are in S_n^{rc} and avoid all the patterns in R. These permutations have been studied and enumerated for various sets of forbidden patterns (see [4], [5], [7]). The result that piqued our interest is [4, Thm 2.17], which we restate here.

Theorem 7 *(Egge) For all* $n \geq 0$*, we have*

$$|S_{2n}^{rc}(123)| = \binom{2n}{n}.$$

Egge's original proof in [4] uses generating functions and the kernal method. He proves a stronger result in [5], which involves a composition of several bijections. We attempted to find a simpler bijection which would take us directly from $S^{rc}_{2n}(123)$ to (say) binary sequences consisting of n 1's and n 0's.

We traced Egge's bijections through to see what happens for certain values of n, hoping that we could look at the resulting binary sequences to find a direct construction to arrive at the result. Despite our best efforts, we were unable to find a simple bijection.

3 Partitions

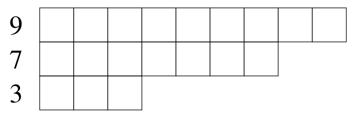
In the remainder of the paper we present several theorems of partitions and their proofs. We also describe our attempts to extend or combine these results to find new proofs of existing theorems. We begin with some definitions and terminology which will be necessary in the rest of the paper.

Definition 8 *A* partition *of a positive integer n is an expression of n as the sum of a sequence of weakly decreasing positive integers, called the* parts *of the partition.*

A partition λ of n with parts $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ is either written like $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s)$ in the form $\lambda_{n_1}^{m_1}\lambda_{n_2}^{m_2}\ldots\lambda_{n_k}^{m_k}$, where each λ_{n_i} is a distinct part in λ with $\lambda_{n_1} > \lambda_{n_2} > \ldots > \lambda_{n_k}$, and there are m_i copies of λ_{n_i}.

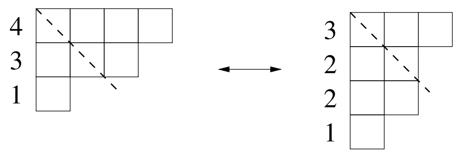
Definition 9 *The* Ferrers diagram *of a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s)$ is a diagram of left-justified boxes with λ_i boxes in the ith row from the top.*

Example 10 *The following is the diagram of $\lambda = (9 \geq 7 \geq 3)$*



Definition 11 *The* conjugate *of a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s)$ is the partition whose Ferrers diagram has λ_i boxes in the ith column from the left.*

Example 12 *The following diagram shows how conjugation can be thought of as reflecting the Ferrers diagram its main diagonal starting in the upper left corner.*



With this terminology in hand, we are ready for our first theorem. The following is just a special case of [2, Cor. 3]. We boil down the proof to a slightly simpler involution that is not a composition of bijections.

Theorem 13 *(Fine) Let $Q(n)$ be the set of partitions of n into distinct parts with odd smallest part. Then $|Q(n)|$ is odd if and only if n is a perfect square.*

Proof. We define an involution, $\kappa : Q(n) \to Q(n)$, which has one fixed point if and only if n is a perfect square (and no fixed points if and only if n is not a perfect square). Let $\lambda \in Q(n)$. Let

$s_1 < s_2 < \ldots < s_k$ be the odd parts of λ and e_1 be the smallest even part of λ. If there is no even part in λ, define $e_1 = \infty$.

Now, let $t_i = s_i - (2i-1)$ for all $i \le k$, and define $t_{k+1} = \infty$. Note that because the partition consists of distinct parts, we have $s_1 \ge 1$, $s_2 \ge 3, \ldots$, and in general $s_i \ge 2i-1$ for all $i \le k$. Thus each t_i is non-negative. Let α be the smallest i such that $t_i > 0$ (alternatively, one can think of α as being the smallest i such that $s_i > 2i-1$). Consider three cases:

(i) $t_\alpha < e_1$. Then split s_α into t_α and $2\alpha - 1$.

(ii) $t_\alpha \ge e_1$ and $e_1 < \infty$. Then combine $s_{\alpha-1}$ with e_1.

(iii) $t_\alpha = e_1 = \infty$. Then do nothing.

Define $\kappa(\lambda)$ to be the result of the above action. We must first show that $\kappa(\lambda) \in Q(n)$.

Suppose λ falls in case (i). Then s_α gets split into t_α (which is even) and $2\alpha - 1$. Since $t_\alpha < e_1$, which is the smallest even in λ, it must be that t_α is the smallest even in $\kappa(\lambda)$, so it is distinct from all other parts. Thus, we only need to show that $2\alpha - 1$ is distinct from all other odds.

If $\alpha = 1$, then $s_1 \ge 3$ gets split into $s_1 - 1$ and 1, and so the smallest (odd) part of $\kappa(\lambda)$ is 1, which is distinct from all other odd parts (since 1 was not part of λ). If $\alpha > 1$, then the odd parts of λ are

$$s_1 = 1 < s_2 = 3 < s_3 = 5 < \ldots < s_{\alpha-1} = 2(\alpha-1) - 1 = 2\alpha - 3 < s_\alpha < \ldots$$

Note that $2\alpha - 3 < 2\alpha - 1$, and $2\alpha - 1 = s_\alpha - t_\alpha < s_\alpha$, so $2\alpha - 1 < s_\alpha$. Therefore, $2\alpha - 1$ is distinct from the odd parts of λ, and so in case (i), $\kappa(\lambda) \in Q(n)$.

Suppose λ falls in case (ii). First, note that $\alpha \ne 1$. Why? If $\alpha = 1$, then $t_\alpha = t_1 = s_1 - 1 < e_1$ (since s_1 is the smallest part of λ), which implies we are in case (i), which is a contradiction.

Because we are in case (ii), we have created an odd part $s_{\alpha-1} + t_\alpha$ which we must verify is distinct from all the other odd parts. Note that

$$s_{\alpha-1} < s_{\alpha-1} + e_1 = 2\alpha - 3 + e_1 < 2\alpha - 1 + e_1 \le 2\alpha - 1 + t_\alpha = s_\alpha,$$

which shows that $s_{\alpha-1} + e_1$ is strictly between $s_{\alpha-1}$ and s_α, which shows that it is distinct from the odds in λ, and so $\kappa(\lambda)$ consists of distinct parts. It remains to show that the smallest part is odd.

Suppose $\alpha = 2$. Then $s_1 = 1$ gets combined with e_1, and $e_1 + 1$ (which is odd) is the smallest part of $\kappa(\lambda)$. If $\alpha \ge 3$, then 1 was the smallest part of λ and remains the smallest part of $\kappa(\lambda)$ (since e_1 gets combined with $s_{\alpha-1}$, which is some part greater than 1 in this case). Thus, we have shown that in case (ii), $\kappa(\lambda) \in Q(n)$. Since case (iii) leaves λ unchanged, we are done.

Now, we show that κ is an involution. Let the odd parts of λ be $s_1 < s_2 < \ldots < s_k$ and the smallest two even parts of λ be $e_1 < e_2$ (if there is only one even part, then $e_2 = \infty$; if there are no even parts, then $e_1 = e_2 = \infty$). Use the same defintion above for t_i and α.

If λ was in case (i) to start, then let $\kappa(\lambda) = \lambda'$ (and e_1' be the smallest even part of λ', etc.). The odd parts of λ' are $1 < 3 < \ldots < 2\alpha - 1 < s_{\alpha+1} < \ldots < s_k$, and $e_1' = t_\alpha$. Note that $t_\alpha' = t_{\alpha+1} \ge t_\alpha = e_1'$, so λ' will fall into case (ii).

Thus, when we take $\kappa(\lambda')$, we will combine e'_1 with $s'_{\alpha-1} = 2\alpha - 1$, which gives us $e'_1 + 2\alpha - 1 = t_\alpha + 2\alpha - 1 = s_\alpha$, so in this case $\kappa(\lambda') = \lambda$.

Suppose λ was in case (ii) to start. Let $\kappa(\lambda) = \lambda''$ (and e''_1 be the smallest even part of λ'' etc.). Then the odd parts of λ'' are

$$1 < 3 < \ldots < 2\alpha - 3 + e_1 < s_\alpha < \ldots < s_k,$$

and the smallest even part of λ'' is $e_2 = e''_1$.

Now, we note that $t''_\alpha = (2\alpha - 3 + e_1) - (2\alpha - 3) = e_1 < e_2 = e''_1$, so we are in case (i) of the mapping. Therefore, $2\alpha - 3 + e_1$ gets split into $2\alpha - 3$ and e_1, which brings us back to our original λ. Therefore, if λ is in case (ii), then $\kappa(\kappa(\lambda)) = \lambda$, and we are done.

Finally, we come to case (iii). This is the only case which leaves λ fixed. Thus, to find the parity of $|Q(n)|$ we only need to consider the elements of $Q(n)$ which fall into case (iii). Suppose λ falls into case (iii). Since $e_1 = \infty$, we know there are no even parts in λ. Since $t_\alpha = \infty$, we know that $t_i = 0$ for all $i \leq k$ (otherwise, $t_i > 0$ for some $i \leq k$, which implies t_α is finite).

Thus, $s_i = 2i - 1$ for all $i \leq k$, which means $\lambda = 1 < 3 < 5 < \ldots < 2k - 1$. From a well known result, then, we have

$$|\lambda| = 1 + 3 + 5 + \ldots + 2k - 1 = k^2,$$

which implies κ has a single fixed point if n is a perfect square and has no fixed points if n is not a perfect square, and the proof is complete.

□

4 Three Proofs of Euler's Theorem

We studied three bijective proofs of the following theorem of Euler's.

Theorem 14 *(Euler) Let $O(n)$ denote the set of partitions of n into odd parts, and let $D(n)$ denote the set of partitions of n into distinct parts. Then for all n, $|O(n)| = |D(n)|$.*

The following proof of Euler's Theorem is credited to Glaisher [8].

Proof. We give a bijection $\phi : D(n) \to O(n)$. For $\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_s) \in D(n)$, let $\lambda_i = 2^{p_i}\mu_i$, where μ_i is odd. Let $\phi(\lambda) = \mu$, where μ has 2^{p_i} parts of μ_i; note that for $i \neq j$ it could be that $\mu_i = \mu_j$, but in that case $p_i \neq p_j$ (or else we would have $\lambda_i = \lambda_j$). Write μ as $\mu_{n_1}^{m_1}\mu_{n_2}^{m_2}\ldots\mu_{n_k}^{m_k}$. Thus $\mu \in O(n)$ because its only parts are the μ_i, which are odd. Since each $m_i = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_q}$ for exactly one sequence of nonnegative integers $i_1 > i_2 > \ldots > i_q$, μ could only be the image under ϕ of some λ with parts of the form $2^{i_t}\mu_i$, so ϕ is one-to-one; furthermore, since such a λ must have distinct parts (different parts would have to have either different odd factors or different even factors, or both), we have a well-defined mapping $\phi^{-1} : O(n) \to D(n)$, so ϕ is onto. □

The following definitions are for the proof of Euler's Theorem given by Pak [9] using a mapping first created by Dyson.

Definition 15 *For any partition λ let $a(\lambda)$ denote the largest part of λ, $l(\lambda)$ denote the number of parts of λ, and $r(\lambda) = a(\lambda) - l(\lambda)$ denote the rank of λ.*

Definition 16 *Let $P(n, r)$ denote the set of partitions of n with rank r, $H(n, r)$ denote the set of partitions of n with rank at most r, and $G(n, r)$ denote the set of partitions of n with rank at least r.*

Definition 17 *Define Dyson's Map $\psi_r : H(n, r + 1) \to G(n + r, r - 1)$: For $\lambda \in H(n, r + 1)$, let $\psi_r(\lambda)$ be the partition whose Ferrers diagram comes from first removing the first column of the Ferrers diagram of λ and then adding a new top row of length $l(\lambda) + r$.*

Definition 18 *Define Iterated Dyson's Map $\zeta : O(n) \to D(n)$: For $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s) \in O(n)$, let $\zeta(\lambda) = \nu^1$, where $\nu^1, \nu^2, \ldots, \nu^s$ are partitions such that ν^s consists of the single part λ_s and $\nu^i = \psi_{\lambda_i}(\nu^{i+1})$.*

Proof. We show that Iterated Dyson's Map ζ is a bijection between $O(n)$ and $D(n)$. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s) \in O(n)$. First note that for all i we have $|\nu^i| = \lambda_i + \ldots + \lambda_s$, so $|\nu^1| = |\lambda| = n$. Now we shall prove by induction that for all i ν^i is a partition into distinct parts with $r(\nu^i) = \lambda_i$ or $\lambda_i - 1$.

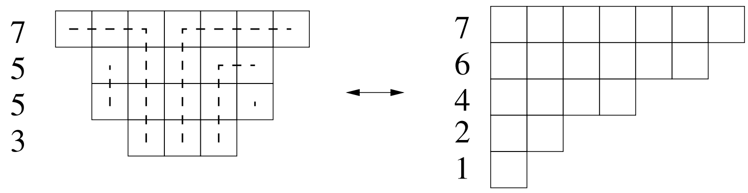
Base Case: ν^s consists of the single part λ_s, so ν^s has distinct parts and rank $\lambda_s - 1$.

Inductive Step: Suppose that $r(\nu^{i+1}) = \lambda_{i+1}$ or $\lambda_{i+1} - 1$ and the parts of ν^{i+1} are distinct. Let ν^i_1 and ν^i_2 denote the length of the first and second rows of ν^i, respectively. Using the facts that $a(\nu^i) = l(\nu^{i+1}) + \lambda_i$, $l(\nu^{i+1}) = a(\nu^{i+1}) - \lambda_{i+1}$ or $a(\nu^{i+1}) - \lambda_{i+1} + 1$, and $\lambda_i \geq \lambda_{i+1}$, we have $\nu^i_1 = a(\nu^i) = l(\nu^{i+1}) + \lambda_i \geq a(\nu^{i+1}) - \lambda_{i+1} + \lambda_i > a(\nu^{i+1}) - 1 = \nu^i_2$. Thus $\nu^i_1 > \nu^i_2$ and the rest of the rows remain distinct because they each lose exactly one box from the removal of the first column. We now show that $r(\nu^i) = \lambda_i$ or $\lambda_i - 1$: $r(\nu^i) = a(\nu^i) - l(\nu^i) = l(\nu^{i+1}) + \lambda_i - l(\nu^i) = \lambda_i$ or $\lambda_i - 1$ since $l(\nu^i) = l(\nu^{i+1})$ or $l(\nu^{i+1}) + 1$, as the parts of ν^{i+1} are distinct so there can be at most a single part of 1.

We now check that $\zeta^{-1} : D(n) \to O(n)$ is well-defined. Starting with $i = 1$, look at $\psi_r^{-1}(\nu^i) = \nu^{i+1}$. Note that $\psi_r^{-1}(\lambda)$ takes the top row of λ, deletes r boxes from it, and places the remaining boxes as the new first column. We just need to show that r is uniquely determined. We need r to be odd for λ_i to be odd since $r = \lambda_i$. We also need $r = \lambda_i = r(\nu^i)$ or $r(\nu^i) + 1 = a(\nu^i) - l(\nu^i)$ or $a(\nu^i) - l(\nu^i) + 1$. Thus r is unique, so $\zeta^{-1} : D(n) \to O(n)$ is well-defined, and thus $\zeta : O(n) \to D(n)$ is a bijection. □

Our final proof is a bijection of Sylvester's [3].

Proof. To apply Sylvester's bijection to a partition into odd parts, first draw the Ferrers diagram of the partition with each row of boxes centered, and then determine the size of each row of the corresponding partition into distinct parts by drawing 'fishhook' lines through the boxes, with the first line going through the central vertical axis and angling out all the way to the right of the top row, and each subsequent line going up through the unmarked vertical column of boxes nearest to the vertical axis and angling to the outside once there are no more unmarked boxes in that column, with the direction of angling alternating each time. A partition into distinct parts can be created by setting each part equal to the number of boxes touched by a fishhook line.



This mapping can be reversed, giving the bijection. We start with the Ferrers graph of some $\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_s) \in D(n)$ and create the centered Ferrers graph of some element of $O(n)$. If s is odd, we insert the boxes of λ_s as a column of dots on the right; otherwise, we insert these boxes as a row of dots on the left. If we have just inserted a part as a fishhook of boxes on the right we insert the next larger part as a fishook of boxes on the left such that the column portion of this fishhook extends one box farther up then the column portion of the previous fishhook. If we have just inserted a part as a fishhook of boxes on the left we insert the next larger part as a fishook of boxes on the right such that the row portion of this fishhook extends one box farther out then the row portion of the previous fishhook. □

We composed pairs of these bijections to create bijections both from $O(n)$ to $O(n)$ and from $D(n)$ to $D(n)$ in the hopes of finding more insights into the structure of these sets. We also composed all three bijections to create new bijections between $O(n)$ and $D(n)$ in the hopes that we could upon inspection give simpler descriptions of these new bijections. Unfortunately, we were unable to draw any sizable conclusions from the experimental data we generated for both of these endeavors.

5 Kolberg's Theorem

The following theorem on the parity of $p(n) = |P(n)|$, where $P(n)$ denotes the set of integer partitions of size n, was first proven in 1959 by Kolberg [6] and later by Newman and by Fabrykowski and Subbarao.

Theorem 19 *(Kolberg) The partition function $p(n)$ takes infinitely many even and odd values.*

Kolberg's proof relies on the well-known recursion relation for $p(n)$ derived by combining two of Euler's results, namely the generating function for $p(n)$ and the pentagonal number theorem. Newman uses the pentagonal number theorem to prove that, for any positive integer m, $p(n)$ fills out at least two different residue classes modulo m infinitely often. Fabrykowski and Subbarao first manipulate the Jacobi triple product identity to obtain a new recursion relation for $p(n)$ and then use this relation to argue about the parity of $p(n)$. We are unaware of any other proofs of Kolberg's Theorem and will now present a direct combinatorial proof.

Proof. Partition conjugation is an involution of $P(n)$ that fixes only partitions whose Ferrers graphs are symmetric about the diagonal from the upper left to lower right. The set $SC(n)$ of self-conjugate partitions of n is in one-to-one correspondence with the set $DO(n)$ of partitions of n into distinct odd parts through the involution that unfolds the Ferrers graph of $\lambda \in SC(n)$ about its axis of symmetry. Thus $p(n)$ has the same parity as $|DO(n)|$, so to prove the theorem we must only show that $|DO(n)|$ changes parity infinitely many times.

For any $\lambda \in DO(n)$ denote the parts of λ by $\lambda_1 > \lambda_2 > \ldots > \lambda_s$. Let $DO_1(n) = \{\lambda \in DO(n) | \lambda_1 - \lambda_2 = 2, \lambda_s > 1\}$. There is a bijection $\phi : DO(n) \to DO(n+1) \backslash DO_1(n+1)$ where for

$\lambda \in DO(n)$, ϕ adds a single part of 1 to the end of λ if $\lambda_s > 1$ and adds 2 to λ_1 while eliminating λ_s if $\lambda_s = 1$. Thus $|DO(n)|$ differs in parity from $|DO(n+1)|$ precisely when $|DO_1(n+1)|$ is odd, so to prove the theorem we can just show that given any integer k we can find an integer $n > k$ such that $|DO_1(n)|$ is odd.

Let $DO_j(n) = \{\lambda \in DO(n) | \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = \ldots = \lambda_j - \lambda_{j+1} = 2, \lambda_s > 1\}$. Then for all positive integers j there is a bijection $\phi_j : DO_j(n) \to DO_j(n+2j+2) \backslash DO_{j+1}(n+2j+2)$ where for $\lambda \in DO(n)$, ϕ_j adds 2 to each of $\lambda_1, \lambda_2, \ldots, \lambda_{j+1}$. Thus $|DO_j(n)|$ differs in parity from $|DO_j(n+2j+2)|$ precisely when $|DO_{j+1}(n+2j+2)|$ is odd.

Given a positive integer k, we now construct an integer $n > k$ such that $|DO_1(n)|$ is odd. Look at $k^2 - 1 = (2k-1) + (2k-3) + \ldots + 3$. Obviously $DO_{k-2}(k^2-1) = \{(2k-1) + (2k-3) + \ldots + 3\}$ and therefore $\left|DO_{k-2}(k^2-1)\right| = 1$. Thus $\left|DO_{k-3}(k^2-1)\right|$ and $\left|DO_{k-3}(k^2-1-2(k-2))\right|$ differ in parity, so we can pick $n_{k-3} \in \left\{k^2-1, k^2-1-2(k-2)\right\}$ such that $|DO_{k-3}(n_{k-3})|$ is odd. Then $|DO_{k-4}(n_{k-3})|$ and $|DO_{k-4}(n_{k-3} - 2(k-3))|$ differ in parity, so we can pick $n_{k-4} \in \{n_{k-3}, n_{k-3} - 2(k-3)\}$ such that $|DO_{k-4}(n_{k-4})|$ is odd. We iterate this process to compute n_1. Then $|DO_1(n_1)|$ is odd, and $n_1 \geq k^2 - 1 - (2(k-2) + 2(k-3) + \ldots + 4) = k^2 - 1 - 2(2 + 3 + \ldots + k - 2) = k^2 - 1 - k(k-3) = 3k - 1 > k$. $\square$

6 Acknowledgements

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References

[1] G. Andrews and K. Eriksson. Integer Partitions. Cambridge University Press, 2004.

[2] C. Bessenrodt, I. Pak. Partition Congruences by Involutions. *Europ. J. Combinatorics*, vol. 25, 2004, 1139-1149.

[3] D.M. Bressoud. Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture. Cambridge University Press, 1999.

[4] E. S. Egge. Restricted Symmetric Permutations. *Ann. Comb.*, 11: 405-434, 2007.

[5] E. S. Egge. Enumerating rc-Invariant Permutations with No Long Decreasing Subsequences. preprint.

[6] O. Kolberg. Note on the parity of the partition function, *Math. Scand.* 7 (1959) 377-378.

[7] D. Lonoff and J. Ostroff. Symmetric permutations avoiding two patterns. preprint.

[8] P. MacMahon. Combinatory Analysis, Vol. 2. Cambridge Univ. Press, 1960.

[9] Pak, Igor. On Fine's Partition Theorems, Dyson, Andrews, and Missed Opportunities. *Math. Intelligencer*, vol. 25, no.1, 2003, 10-16.

[10] The Online Encyclopedia of Integer Sequences, http://www.research.att.com/njas/sequences/.