A Characterization of the Prime Graphs of Solvable Groups

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Advisor: Dr. Thomas Keller

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Definition

Given a finite group $G$, the *prime graph of $G$, $\Gamma_G$, is defined as*

- $V(\Gamma_G) = \{ p \in \mathbb{P} : p \in \pi(G) \}$
- $E(\Gamma_G) = \{ \{ p, q \} : \exists x \in G \text{ with } O(x) = pq \}$.

*Note: the notation $\pi(G)$ refers to the prime divisors of $|G|$.***
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- Research in prime graphs was first motivated by a connection to representation theory discovered by Gruenberg.
- The prime graphs of simple groups are well understood, as well as some graph invariants such as diameter.
Consider $G = S_8$.

- $\pi(S_8) = V(\Gamma_{S_8}) = \{2, 3, 5, 7\}$
- In $S_n$, there is an element of order $pq$ if and only if $p + q \leq n$, since any such element could be decomposed into disjoint $p$ and $q$ cycles.

Therefore, $E(\Gamma_{S_8}) = \\{\{2, 3\}, \{2, 5\}, \{3, 5\}\}$
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- In $S_n$, there is an element of order $pq$ if and only if $p + q \leq n$, since any such element could be decomposed into disjoint $p$ and $q$ cycles. Therefore, $E(\Gamma_{S_8}) = \{\{2, 3\}, \{2, 5\}, \{3, 5\}\}$
Definition: Hall subgroups

Given a set $\pi$ of primes dividing the order of $G$, a $\pi$–Hall subgroup $H$ is a subgroup such that $|H|$ and $[G : H]$ are coprime and $p | |H|$ for each $p \in \pi$. 
Hall Subgroups and Solvable Groups

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Definition: Solvable Group
A group $G$ is solvable if it has $\pi$–Hall subgroups for all subsets of $\pi(G)$, the prime divisors of $|G|$.
Theorem (P. Hall)

If $G$ is a finite solvable group, then every $\pi-$subgroup of $G$ is contained in a Hall $\pi$-subgroup of $G$. Moreover, all Hall $\pi-$subgroups of $G$ are conjugate.

In terms of prime graphs, if $G$ is solvable, then every induced subgraph of $\Gamma_G$ is the prime graph of a subgroup of $G$. 
Theorem

A graph $\Gamma$ is (isomorphic to) the prime graph of some solvable group if and only if $\overline{G}$ is three-colorable and triangle-free.
Three Primes Lemma (Lucido)

Let $G$ be a finite solvable group. If $p$, $q$, $r$ are distinct primes dividing the order of $G$, then $G$ contains an element of order the product of two of these primes.
Proof Part I: Triangle-Free Complements

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- If $G$ is solvable, then for any $p, q, r \in \Gamma_G$, at least one of the edges $pq, qr, rp \in \Gamma_G$. 

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Proof Part I: Triangle-Free Complements

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- If $G$ is solvable, then for any $p, q, r \in \Gamma_G$, at least one of the edges $pq, qr, rp \in \Gamma_G$.
- Put another way, the complement $\overline{\Gamma_G}$ of the prime graph of a solvable group must be triangle free.
Frobenius Groups

Definition: Frobenius Group
A group $G = K \rtimes H$ is a Frobenius group if $H$ is a non identity subgroup such that $H \cap H^g = 1$ for all $g \in G - H$. In this case, we say $H$ is the Frobenius complement and $K$ is the Frobenius kernel of $G$.

Note: $H$ necessarily acts fixed-point freely on $K$ by automorphism.

Example: $D_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$. 
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Definition: 2-Frobenius Group
A group $G = KHF$ is a 2–Frobenius group if $KH$ is a Frobenius group with complement $H$ and $G/K = (HF)/K$ is a Frobenius group with complement $F/K$.

Example: $S_4 = VHF$ where $V \cong \mathbb{Z}_4 \times \mathbb{Z}_3$, and $S_4/V = HF \cong S_3$. 
Frobenius groups have extremely restricted structures; Frobenius kernels are all nilpotent and the Sylow subgroups of Frobenius complements are cyclic or generalized quaternion.
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**Lemma (Lucido)**

If the prime graph of a solvable group $G$ is disconnected, then $G$ is a Frobenius or a 2–Frobenius group.

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Frobenius groups have extremely restricted structures; Frobenius kernels are all nilpotent and the Sylow subgroups of Frobenius complements are cyclic or generalized quaternion.

Lemma (Lucido)

If the prime graph of a solvable group $G$ is disconnected, then $G$ is a Frobenius or a 2−Frobenius group.

Lemma: There are no 3-Frobenius groups.

If $G$ is a group with subgroups $N$ and $L = KHF$ where $L \leq N_G(N)$ and $L$ is a 2-Frobenius group where $KH$ is a Frobenius group with kernel $K$ and $L/K$ is a Frobenius group with kernel $H$, then $K$ cannot act fixed-point freely on $N$. 
Question
If $G$ is solvable, when does $\Gamma_G$ not have an edge? What does this say about the group structure of $G$?

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- If $pq \notin \Gamma_G$, let $H_{pq}$ be a $\{p, q\}$–Hall subgroup. Then $\Gamma_{H_{pq}}$ is comprised of two disconnected points.
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Answer:

- If $pq \notin \Gamma_G$, let $H_{pq}$ be a $\{p, q\}$–Hall subgroup. Then $\Gamma_{H_{pq}}$ is comprised of two disconnected points.
- By Lucido, $H_{pq}$ is a Frobenius or 2–Frobenius group.
Definition: Frobenius Digraph

For a solvable group $G$, the Frobenius digraph, $\Gamma_G$, is constructed from $\Gamma_G$ like so. For two primes $p$ and $q$ adjacent in $\Gamma_G$ and for $H_{pq}$, a Hall $\{p, q\}$—subgroup of $G$,

- if $H_{pq} = KL$ is a Frobenius group with Frobenius kernel $K$, the $\{p, q\}$ edge is directed towards the prime divisor of $K$, and
- if $H_{pq} = KLF$ is a 2-Frobenius group where $H_{pq}/K$ has Frobenius kernel $L$, then the $\{p, q\}$ edge is directed towards the prime divisor of $L$. 
Proof Part II: Three-Colorability of $\Gamma_G$

For $\Gamma_G$, the Frobenius Digraph of some solvable group $G$, we identify the following sets of vertices.
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For $\overrightarrow{\Gamma_G}$, the Frobenius Digraph of some solvable group $G$, we identify the following sets of vertices.

- Let $I$ be the set of vertices with all edges directed into the vertex.

- Let $O$ be the set of vertices with all edges directed away from the vertex.
Proof Part II: Three-Colorability of $\Gamma_G$

For $\Gamma_G$, the Frobenius Digraph of some solvable group $G$, we identify the following sets of vertices.

- Let $I$ be the set of vertices with all edges directed into the vertex.

- Let $O$ be the set of vertices with all edges directed away from the vertex.

- Let $D$ as the set of all vertices with at least one arrow directed towards the vertex and one arrow directed away from the vertex.
No two vertices in the same set can have an edge between them.
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- If two vertices in the set $O$ or $I$ share an edge then there is a double path:

```
    (O) ---- (O)
     |    |    |
     |    |    |
    (O) ---- (O)
```

```
    (I) ---- (I)
     |    |    |
     |    |    |
    (I) ---- (I)
```
No two vertices in the same set can have an edge between them.

- If two vertices in the set $D$ share an edge, then there necessarily exists a three path:
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- If two vertices in the set $D$ share an edge, then there necessarily exists a three path:

```
  o--o--o
   |   |   |
   o   o   o
```

- This is impossible since we cannot have a 3–Frobenius Group.
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- If two vertices in the set $D$ share an edge, then there necessarily exists a three path:

- This is impossible since we cannot have a $3-$Frobenius Group.
- $O$, $I$, and $D$ are independent sets in $\overline{\Gamma_G}$, so $\overline{\Gamma_G}$ is 3-colorable. $\square$
Proof Part III: Converse

**Theorem**

For any 3-colorable, triangle free graph $F$, there exists a finite solvable group $G$ such that $F$ is isomorphic to the complement of $\Gamma_G$. Furthermore, any orientation of $F$ which does not contain a directed 3-path can be realized as a Frobenius digraph of some solvable group.
Using our characterization and several other sources, we were able to show
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- Every finite group is isomorphic to the automorphism group of the prime graph of some solvable group.
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- Every finite group is isomorphic to the automorphism group of the prime graph of some solvable group.
- In the extremal case (i.e. prime graphs with the fewest possible edges), the corresponding groups must have Fitting length between 3 and 5.